What happens to the Ampere's Law ?

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \Rightarrow \nabla \times (\nabla \times \mathbf{A}) = \mu_0 \mathbf{J}$$

$$\Rightarrow \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J}$$

- This is not in a very nice form.
- Ampere's law in terms of **B** seems better

- However, if we can ensure that $\nabla \cdot \mathbf{A} = 0$, we can have it in a nice form.
- This can be done since we know that a $\nabla\lambda$ can be added to **A** without changing **B**

Suppose we start with $\mathbf{A_0}$, such that, $\mathbf{B} = \nabla \times \mathbf{A_0}$ but, $\nabla \cdot \mathbf{A_0} \neq \mathbf{0}$.

Then, $\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \Rightarrow \nabla(\nabla \cdot \mathbf{A_0}) - \nabla^2 \mathbf{A_0} = \mu_0 \mathbf{J}$

Re-define by adding $\nabla\lambda$: $\mathbf{A_0} + \nabla\lambda \equiv \mathbf{A}$ such that $\nabla \cdot \mathbf{A} = \nabla \cdot \mathbf{A_0} + \nabla^2 \lambda = 0$

Then $\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \Rightarrow \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J} \Rightarrow -\nabla^2 \mathbf{A} = \mu_0 \mathbf{J}$

Thus, one can always redefine the vector potential such that $\nabla \cdot \mathbf{A} = 0$

Poisson'e equation

Recall: $\nabla^2 V = -\frac{\rho}{\epsilon_0}$ (Poisson's Equation)

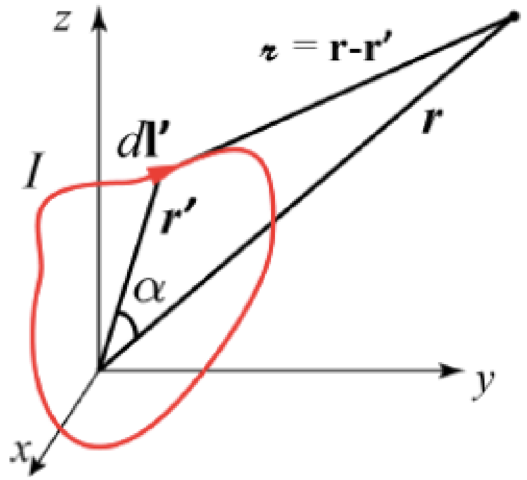
The solution is: $V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{\mathscr{r}} d\tau'$

So, $\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')}{\mathscr{r}} d\tau'$ This is simpler than Biot-Savart Law.

For surface current: $\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{K}(\mathbf{r}')}{\mathscr{r}} da'$

For line current: $\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} I \int \frac{d\mathbf{l}'}{\mathscr{r}}$

Magnetic potential



Source coordinates: (r', θ', ϕ')
Observation point coordinates: (r, θ, ϕ)
Angle between $\mathbf{r}$ and $\mathbf{r}'$: α

$$\mathscr{r}^2 = r^2 + r'^2 - 2rr'\cos\alpha$$

$$\mathscr{r}^2 = r^2\left[1 + \left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\alpha\right]$$

$$\frac{1}{\mathscr{r}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\alpha)$$

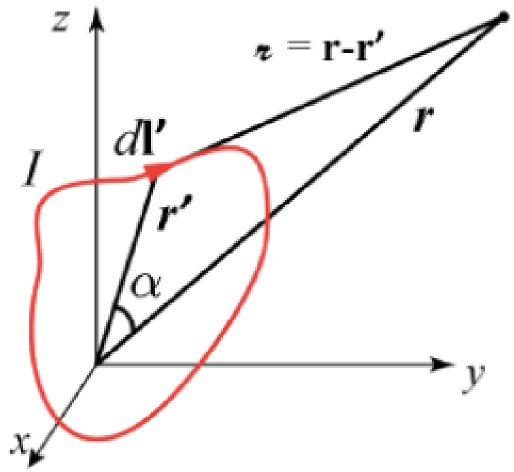
$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\oint\frac{d\mathbf{l}'}{\mathscr{r}}$$

$$= \frac{\mu_0 I}{4\pi}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\oint(r')^n\, P_n(\cos\alpha)d\mathbf{l}'$$

$$= \frac{\mu_0 I}{4\pi}\frac{1}{r}\oint d\mathbf{l}' + \frac{\mu_0 I}{4\pi}\frac{1}{r^2}\oint r'\cos\alpha\, d\mathbf{l}' + \frac{\mu_0 I}{4\pi}\frac{1}{r^3}\oint(r')^2\left(\frac{3}{2}\cos\alpha - \frac{1}{2}\right)d\mathbf{l}' + \cdots$$

Monopole potential ($1/r$ dependence) | Dipole potential ($1/r^2$ dependence) | Quadrupole potential ($1/r^3$ dependence)

Magnetic potential



Monopole potential

$$\mathbf{A}_{\text{mono}}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\frac{1}{r}\oint d\mathbf{l}' \quad = 0$$

Dipole potential

$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\frac{1}{r^2}\oint r'\cos\alpha\, d\mathbf{l}' \;= \frac{\mu_0 I}{4\pi}\frac{1}{r^2}\oint (\hat{\mathbf{r}}\cdot\mathbf{r}')d\mathbf{l}'$$

$$= -\frac{\mu_0 I}{4\pi}\frac{1}{r^2}\int \nabla'(\hat{\mathbf{r}}\cdot\mathbf{r}')\times d\mathbf{a}'$$

[Corollary of Stokes Theorem: $\oint_{path} T d\mathbf{l} = -\int_{Surf} \nabla T \times d\mathbf{a}$]

$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = -\frac{\mu_0 I}{4\pi}\frac{1}{r^2}\int \hat{\mathbf{r}}\times d\mathbf{a}' \;= \frac{\mu_0}{4\pi}\frac{1}{r^2}\left(I\int d\mathbf{a}'\right)\times\hat{\mathbf{r}}$$

$$= \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\hat{\mathbf{r}}}{r^2}$$

$$\boxed{\mathbf{m} \equiv I\int d\mathbf{a}'}$$

Magnetic dipole moment

Magnetic field due to magnetic dipole

$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\hat{\mathbf{r}}}{r^2}$$

Take $\mathbf{m} = m\,\hat{\mathbf{z}}$

$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^2}\hat{\boldsymbol{\phi}}$$

$$\mathbf{B}_{\text{dip}}(\mathbf{r}) = \nabla\times\mathbf{A}_{\text{dip}}(\mathbf{r})$$

$$= \frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^3}\left(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}\right)$$

Recall

$\mathbf{p} = p\hat{\mathbf{z}}$

$$\mathbf{E}_{\text{dip}}(\mathbf{r}) = \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\theta})$$

Polarization & Magnetization

What is Polarization? - dipole moment per unit volume

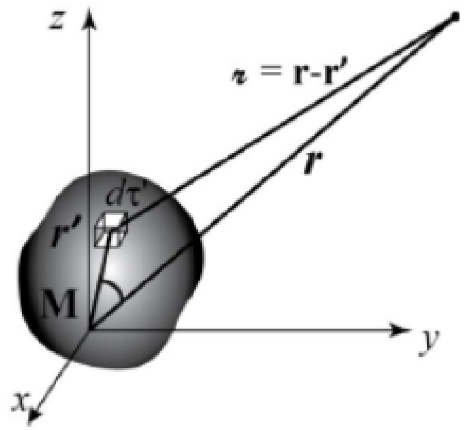
- The dipole moment is caused either by stretch of an atom/molecule or by rotation of polar molecules
- Polarization in the direction parallel to the applied electric field

What is Magnetization? - magnetic dipole moment per unit volume

- The magnetic dipole moment is caused by electric charges in motion: (i) electrons orbiting around nuclei & (ii) electrons spinning about their own axes.
- In some material, magnetization is in the direction parallel to **B** (Paramagnets).
- In some other material, magnetization is opposite to **B** (Diamagnets).
- In other, there can be magnetization even in the absence of **B** (Ferromagnets).

Field of a magnetized object

The Field of a Magnetized Object:



$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\hat{\mathbf{r}}}{r^2} = \frac{\mu_0}{4\pi}\int_{vol}\frac{\mathbf{M}(\mathbf{r}')\times\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}d\tau'$$

$$= \frac{\mu_0}{4\pi}\int_{vol}\left[\mathbf{M}(\mathbf{r}')\times\nabla'\left(\frac{1}{\mathscr{r}}\right)\right]d\tau'$$

[Using $\nabla'\left(\frac{1}{\mathscr{r}}\right) = \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$]

$$= \frac{\mu_0}{4\pi}\int_{vol}\frac{1}{\mathscr{r}}[\nabla'\times\mathbf{M}(\mathbf{r}')]d\tau' - \frac{\mu_0}{4\pi}\int_{vol}\nabla'\times\left[\frac{\mathbf{M}(\mathbf{r}')}{\mathscr{r}}\right]d\tau'$$

[Using $\nabla\times(f\mathbf{A}) = f(\nabla\times\mathbf{A}) - \mathbf{A}\times(\nabla f)$]

$$= \frac{\mu_0}{4\pi}\int_{vol}\frac{1}{\mathscr{r}}[\nabla'\times\mathbf{M}(\mathbf{r}')]d\tau' + \frac{\mu_0}{4\pi}\oint_{surf}\frac{1}{\mathscr{r}}[\mathbf{M}(\mathbf{r}')\times d\mathbf{a}']$$

$\int_{vol}(\nabla\times\mathbf{V})d\tau = -\oint_{surf}\mathbf{V}\times d\mathbf{a}$]

$$= \frac{\mu_0}{4\pi}\int_{vol}\frac{\mathbf{J}_b(\mathbf{r}')}{\mathscr{r}}d\tau' + \frac{\mu_0}{4\pi}\oint_{surf}\frac{\mathbf{K}_b(\mathbf{r}')}{\mathscr{r}}d\mathbf{a}'$$

$\mathbf{J}_b(\mathbf{r}') = \nabla'\times\mathbf{M}(\mathbf{r}')$ Volume current

$\mathbf{K}_b(\mathbf{r}') = \mathbf{M}(\mathbf{r}')\times\hat{\mathbf{n}}$ Surface current

Ampere's law in magnetized material:

$\mathbf{J}_b(\mathbf{r}') = \nabla' \times \mathbf{M}(\mathbf{r}')$

Volume current

$\mathbf{K}_b(\mathbf{r}') = \mathbf{M}(\mathbf{r}') \times \hat{\mathbf{n}}$

Surface current

Total volume current is

$$\mathbf{J} = \mathbf{J}_f + \mathbf{J}_b$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \quad = \mu_0(\mathbf{J}_f + \mathbf{J}_b) \quad = \mu_0(\mathbf{J}_f + \nabla \times \mathbf{M})$$

Ampere's law in magnetized material:

$\mathbf{J}_b(\mathbf{r}') = \nabla' \times \mathbf{M}(\mathbf{r}')$

Volume current

$\mathbf{K}_b(\mathbf{r}') = \mathbf{M}(\mathbf{r}') \times \hat{\mathbf{n}}$

Surface current

Total volume current is

$$\mathbf{J} = \mathbf{J}_f + \mathbf{J}_b$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \quad = \mu_0(\mathbf{J}_f + \mathbf{J}_b) \quad = \mu_0(\mathbf{J}_f + \nabla \times \mathbf{M})$$

$$\nabla \times \left(\frac{\mathbf{B}}{\mu_0} - \mathbf{M}\right) = \mathbf{J}_f$$

Define: $\mathbf{H} \equiv \dfrac{\mathbf{B}}{\mu_0} - \mathbf{M}$ $\qquad \mathbf{B} \equiv \mu_0(\mathbf{H} + \mathbf{M})$

$\nabla \times \mathbf{H} = \mathbf{J}_f$ Ampere's law in magnetized material (differential form)

$\oint \mathbf{H} \cdot d\mathbf{l} = I_{f\text{enc}}$ Ampere's law in magnetized material (integral form)

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Magnetic Susceptibility and Permeability

$$\mathbf{M} = \chi_m \mathbf{H}$$

χ_m is called the magnetic susceptibility

χ_m is a dimensionless quantity

χ_m is positive for paramagnetic and negative for diamagnets

The magnetic field thus becomes

$$\mathbf{B} \equiv \mu_0(\mathbf{H} + \mathbf{M}) = \mu_0(\mathbf{H} + \chi_m \mathbf{H}) = \mu_0(1 + \chi_m)\mathbf{H}$$

So, $\mathbf{B} \equiv \mu \mathbf{H}$

$$\mu \equiv \mu_0(1 + \chi_m)$$

is called the permeability of the material