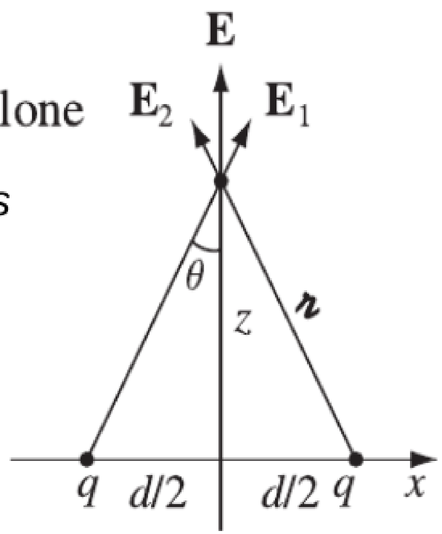
Calculating electric field: two point charges (++)

Find the electric field a distance *z* above the midpoint between two equal charges (*q*), a distance *d* apart.

→ Let $\mathbf{E}_1$ be the field of the left charge alone, and $\mathbf{E}_2$ that of the right charge alone

Vectorially adding them the horizontal components cancel and the vertical components add up.



$$E_z = 2\frac{1}{4\pi\epsilon_0}\frac{q}{\mathcal{r}^2}\cos\theta.$$

Here $\mathcal{r} = \sqrt{z^2 + (d/2)^2}$ and $\cos\theta = z/\mathcal{r}$, so

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{2qz}{\left[z^2 + (d/2)^2\right]^{3/2}}\hat{\mathbf{z}}.$$

Check: When $z \gg d$ you're so far away that it just looks like a single charge $2q$, so the field should reduce to $\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{2q}{z^2}\hat{\mathbf{z}}$. And it *does* (just set $d \to 0$ in the formula).

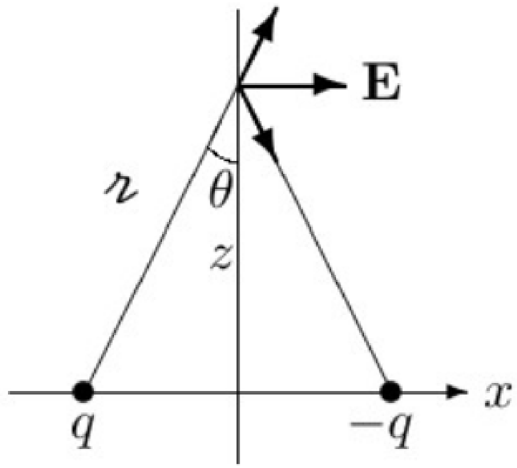
Calculating electric field: two point charges (+ -)

What happens when charges are of same magnitude but opposite sign?

Now the vertical components cancel, resulting in

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} 2 \frac{q}{\mathscr{r}^2} \sin\theta\, \hat{\mathbf{x}}, \text{ or}$$

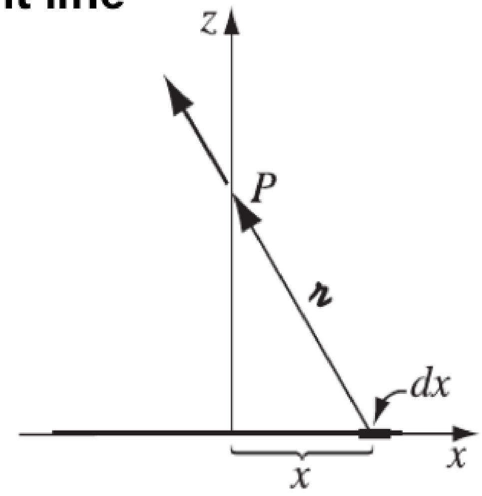
$$\boxed{\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{qd}{\left(z^2 + \left(\frac{d}{2}\right)^2\right)^{3/2}} \hat{\mathbf{x}}.}$$



From far away, $(z \gg d)$, the field goes like $\mathbf{E} \approx \frac{1}{4\pi\epsilon_0} \frac{qd}{z^3} \hat{\mathbf{z}}$

Calculating electric field: line charge

Find the electric field a distance *z* above the midpoint of a straight line segment of length *2L* carrying a uniform line charge λ



$$\mathbf{r} = z\,\hat{\mathbf{z}}, \quad \mathbf{r}' = x\,\hat{\mathbf{x}}, \quad dl' = dx;$$

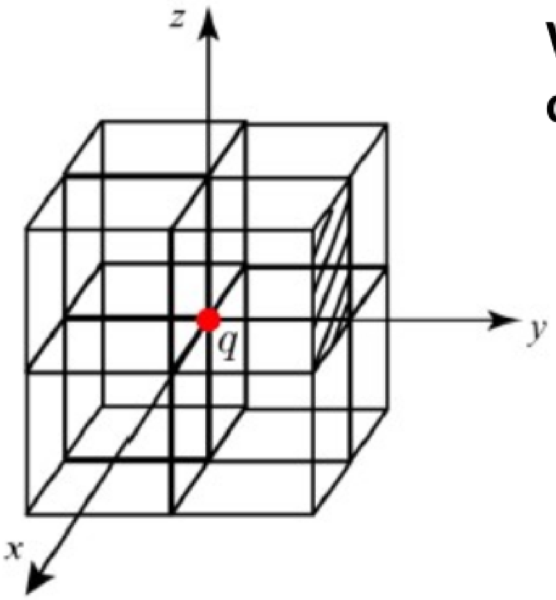
$$\boldsymbol{\mathscr{r}} = \mathbf{r} - \mathbf{r}' = z\,\hat{\mathbf{z}} - x\,\hat{\mathbf{x}}, \quad \mathscr{r} = \sqrt{z^2 + x^2}, \quad \hat{\boldsymbol{\mathscr{r}}} = \frac{\boldsymbol{\mathscr{r}}}{\mathscr{r}} = \frac{z\,\hat{\mathbf{z}} - x\,\hat{\mathbf{x}}}{\sqrt{z^2 + x^2}}.$$

$$\begin{aligned}
\mathbf{E} &= \frac{1}{4\pi\epsilon_0} \int_{-L}^{L} \frac{\lambda}{z^2 + x^2} \frac{z\,\hat{\mathbf{z}} - x\,\hat{\mathbf{x}}}{\sqrt{z^2 + x^2}}\, dx \\
&= \frac{\lambda}{4\pi\epsilon_0} \left[z\,\hat{\mathbf{z}} \int_{-L}^{L} \frac{1}{(z^2 + x^2)^{3/2}}\, dx - \hat{\mathbf{x}} \int_{-L}^{L} \frac{x}{(z^2 + x^2)^{3/2}}\, dx \right] \\
&= \frac{\lambda}{4\pi\epsilon_0} \left[z\,\hat{\mathbf{z}} \left(\frac{x}{z^2\sqrt{z^2 + x^2}} \right) \Bigg|_{-L}^{L} - \hat{\mathbf{x}} \left(-\frac{1}{\sqrt{z^2 + x^2}} \right) \Bigg|_{-L}^{L} \right] \\
&= \frac{1}{4\pi\epsilon_0} \frac{2\lambda L}{z\sqrt{z^2 + L^2}}\,\hat{\mathbf{z}}.
\end{aligned}$$

For points far from the line ($z \gg L$),

$$E \cong \frac{1}{4\pi\epsilon_0} \frac{2\lambda L}{z^2}.$$

Calculating flux: point charge



What is the flux through the shaded face of the cube due to the charge q at the corner?

$$\int_{surf} \mathbf{E} \cdot d\mathbf{a} \; ??$$

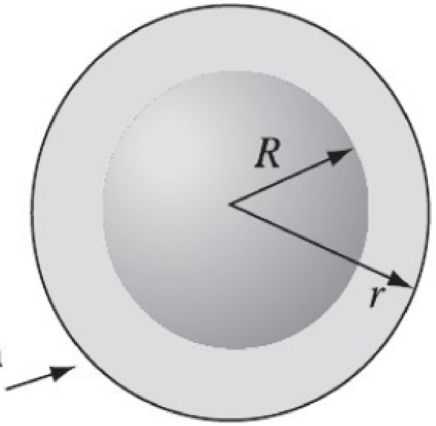
$$24 \int_{surf} \mathbf{E} \cdot d\mathbf{a} = \frac{q}{\epsilon_0}$$

$$\int_{surf} \mathbf{E} \cdot d\mathbf{a} = \frac{1}{24} \frac{q}{\epsilon_0}$$

$$\oint_{surf} \mathbf{E} \cdot d\mathbf{a} = \frac{Q_{\text{enc}}}{\epsilon_0}$$

Calculating electric field:charge distributed in sphere

Find the electric field outside a uniformly charged solid sphere of radius *R* and total charge *q*.



Imagine a spherical surface at radius $r > R$; $Q_{\text{enc}} = q$

$$\oint_S \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{\text{enc}}$$

symmetry allows us to extract **E** from under the integral sign: **E** certainly points radially outward, as does $d\mathbf{a}$, so we can drop the dot product,
and the *magnitude* of **E** is constant over the Gaussian surface, so it comes outside the integral:

$$\int_S \mathbf{E} \cdot d\mathbf{a} = \int_S |\mathbf{E}|\, da = |\mathbf{E}| \int_S da = |\mathbf{E}|\, 4\pi r^2 = \frac{1}{\epsilon_0} q$$

Therefore,

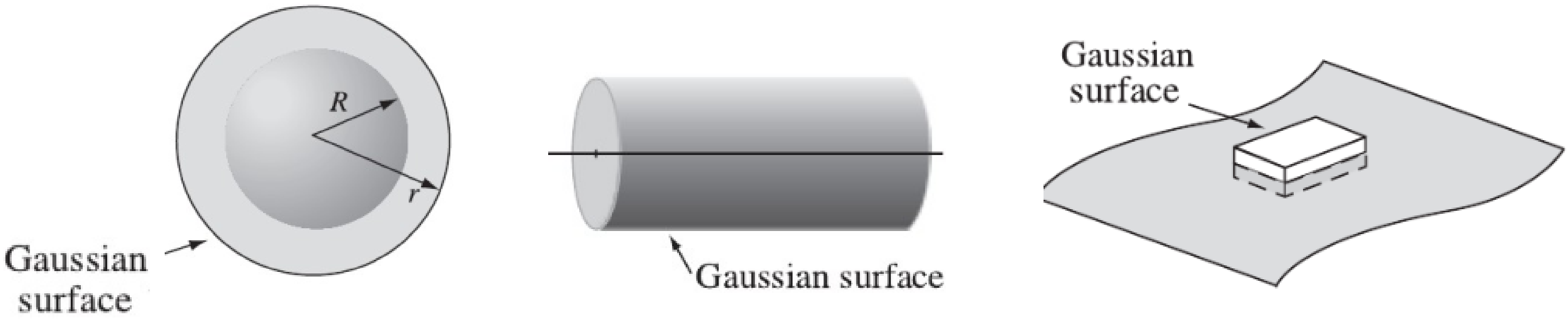
$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{\mathbf{r}}$$

Notice a remarkable feature of this result: The field outside the sphere is exactly *the same as it would have been if all the charge had been concentrated at the center.*

How to choose Gaussian surface?

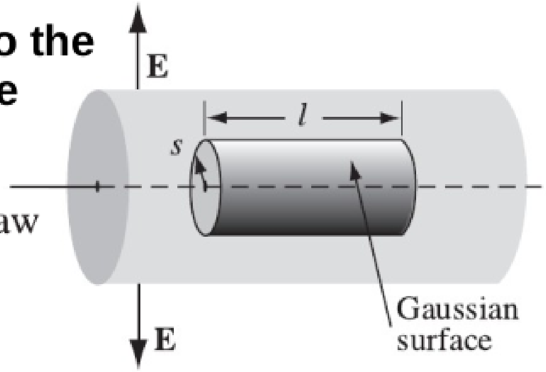
Gauss's law is always *true*, but it is not always *useful*. If ρ had not been uniform (or, at any rate, not spherically symmetrical), or if I had chosen some other shape for my Gaussian surface, it would still have been true that the flux of $\mathbf{E}$ is q/ϵ_0, but $\mathbf{E}$ would not have pointed in the same direction as $d\mathbf{a}$, and its magnitude would not have been constant over the surface, and without that I cannot get $|\mathbf{E}|$ outside of the integral. *Symmetry is crucial* to this application of Gauss's law.

1. *Spherical symmetry.* Make your Gaussian surface a concentric sphere.
2. *Cylindrical symmetry.* Make your Gaussian surface a coaxial cylinder
3. *Plane symmetry.* Use a Gaussian "pillbox" that straddles the surface



Cylindrical Gaussian surface

A long cylinder carries a charge density that is proportional to the distance from the axis: $\rho = ks$, for some constant *k*. Find the electric field inside the cylinder.



Draw a Gaussian cylinder of length l and radius s. For this surface, Gauss's law states:

$$\oint_S \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{\text{enc}}.$$

The enclosed charge is

$$Q_{\text{enc}} = \int \rho \, d\tau = \int (ks')(s' \, ds' \, d\phi \, dz) = 2\pi k l \int_0^s s'^2 \, ds' = \tfrac{2}{3}\pi k l s^3.$$

⟶ integrated ϕ from 0 to 2π, dz from 0 to l.

Now, symmetry dictates that $\mathbf{E}$ must point radially outward, so for the curved portion of the Gaussian cylinder we have:

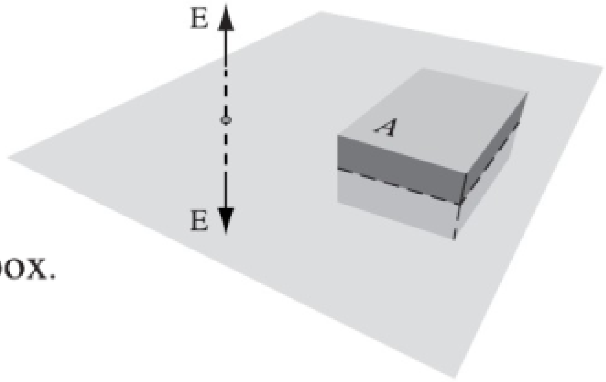
$$\int \mathbf{E} \cdot d\mathbf{a} = \int |\mathbf{E}| \, da = |\mathbf{E}| \int da = |\mathbf{E}| \, 2\pi s l,$$

Hence, $|\mathbf{E}| \, 2\pi s l = \frac{1}{\epsilon_0} \frac{2}{3} \pi k l s^3 \longrightarrow \mathbf{E} = \frac{1}{3\epsilon_0} k s^2 \hat{\mathbf{s}}.$

Note that, the two ends of the surface do not contribute anything since the electric field is perpendicular to the surface.

Pillbox Gaussian Surface

An infinite plane carries a uniform surface charge density σ. Find its electric field.



Apply Gauss's law to this surface:

$$\oint \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{\text{enc}}.$$

In this case, $Q_{\text{enc}} = \sigma A$, where A is the area of the lid of the pillbox.

$\mathbf{E}$ points away from the plane (upward for points above, downward for points below).

So the top and bottom surfaces yield $$\int \mathbf{E} \cdot d\mathbf{a} = 2A|\mathbf{E}|,$$

whereas the sides contribute nothing. Thus

$$2A\,|\mathbf{E}| = \frac{1}{\epsilon_0} \sigma A$$

Hence, $$\mathbf{E} = \frac{\sigma}{2\epsilon_0} \hat{\mathbf{n}}$$ where $\hat{\mathbf{n}}$ is a unit vector pointing away from the surface.

More properties of electric field

- Is electric field conservative?
- How do we define potential?
- To check if ***E*** is conservative, we have to find curl of ***E***, or closed path line integral. Remember Stoke's theorem?

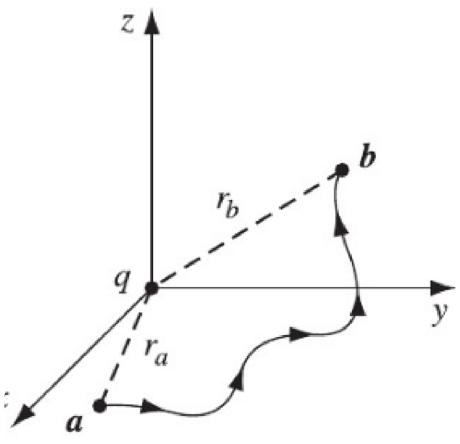
Start with simplest possible configuration: *Electric field due to a single charge, q*

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}$$

If we calculate the line integral of this field from point **a** to point **b**, $\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E}\cdot d\mathbf{l}$.

In spherical coordinates, $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}$, so

$$\mathbf{E}\cdot d\mathbf{l} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}dr.$$

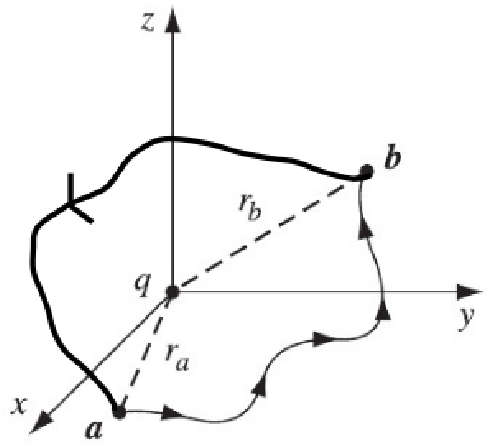


Therefore,

$$\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E}\cdot d\mathbf{l} = \frac{1}{4\pi\epsilon_0}\int_{\mathbf{a}}^{\mathbf{b}}\frac{q}{r^2}dr = \frac{-1}{4\pi\epsilon_0}\frac{q}{r}\bigg|_{r_a}^{r_b} = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_a}-\frac{q}{r_b}\right),$$

Curl of Electric field

The integral around a *closed* path is evidently zero (for then $r_a = r_b$):



$$\oint \mathbf{E} \cdot d\mathbf{l} = 0,$$

and hence, applying Stokes' theorem,

$$\nabla \times \mathbf{E} = \mathbf{0}.$$

(True for any static charge distribution)

Moreover, if we have many charges, the principle of superposition states that total field is a vector sum of their individual fields:

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 + \ldots,$$

so

$$\nabla \times \mathbf{E} = \nabla \times (\mathbf{E}_1 + \mathbf{E}_2 + \ldots) = (\nabla \times \mathbf{E}_1) + (\nabla \times \mathbf{E}_2) + \ldots = \mathbf{0}.$$

→ ***For a vector to represent an electric field, its curl must be zero.***

Electric potential

Since the line integral is independent of the path and only depends on the end points, we can define a scalar function,

$$V(\mathbf{r}) \equiv -\int_{\mathcal{O}}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{l}.$$

Here $\mathcal{O}$ is some standard reference point on which we have agreed beforehand; V then depends only on the point $\mathbf{r}$. It is called the **electric potential.**

The potential *difference* between two points **a** and **b** is

$$\begin{aligned} V(\mathbf{b}) - V(\mathbf{a}) &= -\int_{\mathcal{O}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l} + \int_{\mathcal{O}}^{\mathbf{a}} \mathbf{E} \cdot d\mathbf{l} \\ &= -\int_{\mathcal{O}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l} - \int_{\mathbf{a}}^{\mathcal{O}} \mathbf{E} \cdot d\mathbf{l} = -\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l}. \end{aligned}$$

Gradient of a scalar always has zero curl.

Electric potential

The potential *difference* between two points **a** and **b** is

$$V(\mathbf{b}) - V(\mathbf{a}) = -\int_{\mathcal{O}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l} + \int_{\mathcal{O}}^{\mathbf{a}} \mathbf{E} \cdot d\mathbf{l}$$

$$= -\int_{\mathcal{O}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l} - \int_{\mathbf{a}}^{\mathcal{O}} \mathbf{E} \cdot d\mathbf{l} = -\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l}.$$

We know from the fundamental theorem of gradients,

$$V(\mathbf{b}) - V(\mathbf{a}) = \int_{\mathbf{a}}^{\mathbf{b}} (\nabla V) \cdot d\mathbf{l}$$

Hence,

$$\int_{\mathbf{a}}^{\mathbf{b}} (\nabla V) \cdot d\mathbf{l} = -\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l}.$$

Since, finally, this is true for *any* points **a** and **b**, the integrands must be equal:

$$\boxed{\mathbf{E} = -\nabla V.}$$

Gradient of a scalar always has zero curl.