Poisson and Laplace equation

Electric field can be written as the gradient of potential: $\mathbf{E} = -\nabla V.$

→ Looking at the divergence and curl of an electric field, $\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$ and $\nabla \times \mathbf{E} = \mathbf{0}$

In terms of the potential: $\nabla \cdot \mathbf{E} = \nabla \cdot (-\nabla V) = -\nabla^2 V$

Then from Gauss' law:

$$\boxed{\nabla^2 V = -\frac{\rho}{\epsilon_0}.}$$ → ***Poisson's Equation***

→ In regions with no charge, i.e., if $\rho = 0$

$$\nabla^2 V = 0.$$ → ***Laplace's Equation***

The curl of electric field is also consistent since

$$\nabla \times \mathbf{E} = \nabla \times (-\nabla V) = \mathbf{0}.$$

Electric potential

Knowing the electric field, one can calculate the potential.

$$\mathbf{E} = (1/4\pi\epsilon_0)(q/r^2)\,\hat{\mathbf{r}}$$

$$d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}$$

Then,

$$\mathbf{E}\cdot d\mathbf{l} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\,dr.$$

Setting the reference point at infinity, the potential of a point charge q at the origin is

$$V(r) = -\int_{\mathcal{O}}^{\mathbf{r}} \mathbf{E}\cdot d\mathbf{l} = \frac{-1}{4\pi\epsilon_0}\int_{\infty}^{r}\frac{q}{r'^2}\,dr' = \frac{1}{4\pi\epsilon_0}\frac{q}{r'}\bigg|_{\infty}^{r} = \frac{1}{4\pi\epsilon_0}\frac{q}{r}.$$

In general, the potential of a point charge q is

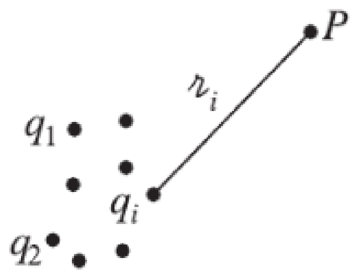
$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{\mathscr{r}}$$

where $\mathscr{r}$, as always, is the distance from q to $\mathbf{r}$

Electric potential

Following superposition principle, for a collection of charges,

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{\mathscr{r}_i}$$



For a continuous charge distribution,

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{1}{\mathscr{r}}dq\,.$$

→ For a line, surface and volume charge distribution respectively,

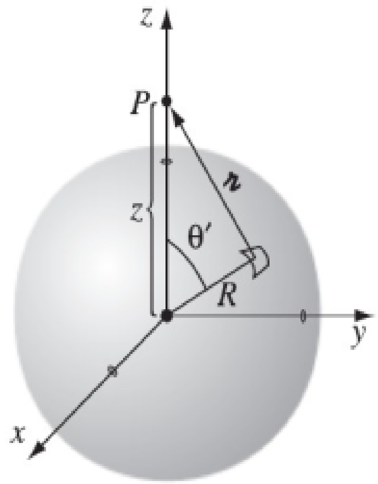
$$V = \frac{1}{4\pi\epsilon_0}\int \frac{\lambda(\mathbf{r}')}{\mathscr{r}}dl' \qquad V = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma(\mathbf{r}')}{\mathscr{r}}da' \qquad V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{\mathscr{r}}d\tau'$$

Electric potential: Charged sphere

Find the electric potential inside and outside a uniformly charged spherical shell of radius R

We'll do it using the charge distribution.

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma}{\mathscr{r}}\,da'$$



We might as well set the point P on the z axis and use the law of cosines to express $\mathscr{r}$:

$$\mathscr{r}^2 = R^2 + z^2 - 2Rz\cos\theta'.$$

An element of surface area on the sphere is $R^2 \sin\theta'\, d\theta'\, d\phi'$, so

$$\begin{aligned}
4\pi\epsilon_0 V(z) &= \sigma \int \frac{R^2 \sin\theta'\, d\theta'\, d\phi'}{\sqrt{R^2+z^2-2Rz\cos\theta'}} \\
&= 2\pi R^2 \sigma \int_0^{\pi} \frac{\sin\theta'}{\sqrt{R^2+z^2-2Rz\cos\theta'}}\, d\theta' \\
&= 2\pi R^2 \sigma \left(\frac{1}{Rz}\sqrt{R^2+z^2-2Rz\cos\theta'}\right)\Bigg|_0^{\pi} \\
&= \frac{2\pi R\sigma}{z}\left(\sqrt{R^2+z^2+2Rz} - \sqrt{R^2+z^2-2Rz}\right) \\
&= \frac{2\pi R\sigma}{z}\left[\sqrt{(R+z)^2} - \sqrt{(R-z)^2}\right].
\end{aligned}$$

Electric potential: Charged sphere

At this stage, we must be very careful to take the *positive* root. For points *outside* the sphere, z is greater than R, and hence $\sqrt{(R-z)^2} = z - R$; for points *inside* the sphere, $\sqrt{(R-z)^2} = R - z$. Thus,

$$V(z) = \frac{R\sigma}{2\epsilon_0 z}[(R+z) - (z-R)] = \frac{R^2\sigma}{\epsilon_0 z}, \qquad \text{outside;}$$

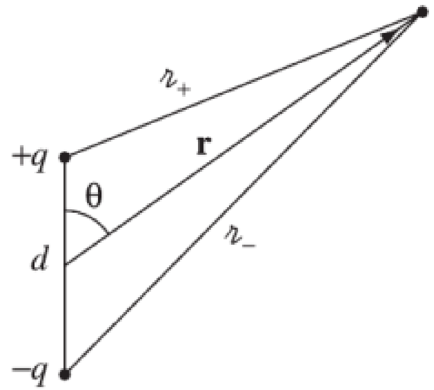
$$V(z) = \frac{R\sigma}{2\epsilon_0 z}[(R+z) - (R-z)] = \frac{R\sigma}{\epsilon_0}, \qquad \text{inside.}$$

In terms of r and the total charge on the shell, $q = 4\pi R^2\sigma$,

$$V(r) = \begin{cases} \dfrac{1}{4\pi\epsilon_0}\dfrac{q}{r} & (r \geq R), \\[2ex] \dfrac{1}{4\pi\epsilon_0}\dfrac{q}{R} & (r \leq R). \end{cases}$$

Electric dipole

An electric dipole with equal and opposite charges separated by a distance *d*. Calculate the potential at a distance ***r***.



Let $\mathscr{r}_-$ be the distance from $-q$ and $\mathscr{r}_+$ the distance from $+q$

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\mathscr{r}_+} - \frac{q}{\mathscr{r}_-}\right)$$

We have,

$$\mathscr{r}_\pm^2 = r^2 + (d/2)^2 \mp r d\cos\theta = r^2\left(1 \mp \frac{d}{r}\cos\theta + \frac{d^2}{4r^2}\right)$$

For $r \gg d$, binomial expansion yields

$$\frac{1}{\mathscr{r}_\pm} \cong \frac{1}{r}\left(1 \mp \frac{d}{r}\cos\theta\right)^{-1/2} \cong \frac{1}{r}\left(1 \pm \frac{d}{2r}\cos\theta\right)$$

Then

$$\frac{1}{\mathscr{r}_+} - \frac{1}{\mathscr{r}_-} \cong \frac{d}{r^2}\cos\theta$$

Hence,

$$V(\mathbf{r}) \cong \frac{1}{4\pi\epsilon_0}\frac{q d\cos\theta}{r^2}.$$

Potential due to electric dipole

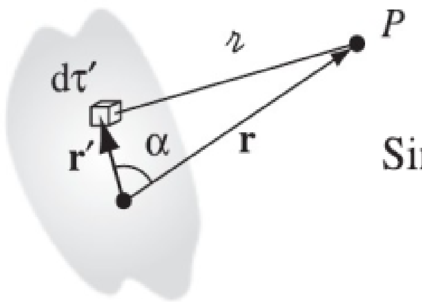
For a given charge distribution, the most dominant contribution to potential comes from the monopole term.

$$V_{\text{mon}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r}$$

where, $Q = \int \rho \, d\tau$

For a point charge at origin, this is the exact contribution. All higher order contribution vanish.

⟶ If the total charge happens to be zero, then most dominant contribution comes from the nno-zero dipole term.



$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\int r' \cos\alpha \, \rho(\mathbf{r}') \, d\tau'$$

Since α is the angle between $\mathbf{r}'$ and $\mathbf{r}$

$$r' \cos\alpha = \hat{\mathbf{r}} \cdot \mathbf{r}'$$

Then

$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\hat{\mathbf{r}} \cdot \int \mathbf{r}' \rho(\mathbf{r}') \, d\tau'$$

Dipole moment

Define dipole moment:

$$\mathbf{p} \equiv \int \mathbf{r}' \rho(\mathbf{r}')\, d\tau'$$

Then we can write,

$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{\mathbf{p} \cdot \hat{\mathbf{r}}}{r^2}$$

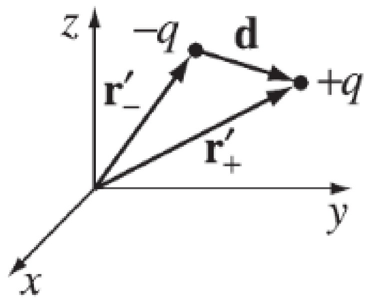
For a collection of point charges,

$$\mathbf{p} = \sum_{i=1}^{n} q_i \mathbf{r}'_i$$

For a physical dipole it reduces to:

$$\mathbf{p} = q\mathbf{r}'_+ - q\mathbf{r}'_- = q(\mathbf{r}'_+ - \mathbf{r}'_-) = q\mathbf{d}$$

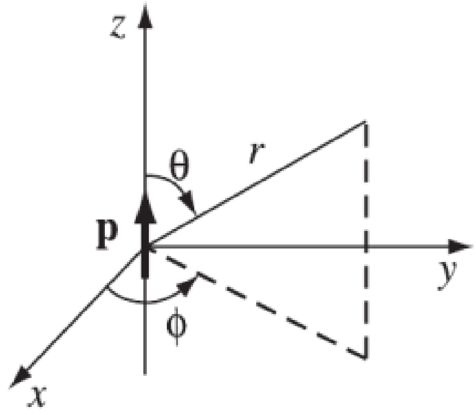
→ valid only for $r \gg d$



Electric field due to dipole

We have found out the potential in terms of dipole moment:

$$V_{\mathrm{dip}}(r,\theta) = \frac{\hat{\mathbf{r}}\cdot\mathbf{p}}{4\pi\epsilon_0 r^2} = \frac{p\cos\theta}{4\pi\epsilon_0 r^2}$$



To get the field, we take the negative gradient of V:

$$E_r = -\frac{\partial V}{\partial r} = \frac{2p\cos\theta}{4\pi\epsilon_0 r^3},$$

$$E_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta} = \frac{p\sin\theta}{4\pi\epsilon_0 r^3},$$

$$E_\phi = -\frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi} = 0.$$

Thus,

$$\mathbf{E}_{\mathrm{dip}}(r,\theta) = \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}).$$