Atomic dipole

When neutral atom is placed in an electric field:

→ The positive and negative charge cores are separated; the nucleus pushed in the direction of the field and the electrons the opposite way.

→ The positive and negative core attract each other; it holds the atom together.

→ The atom is polarised and now has a tiny dipole moment pointing in the same direction as the external electric field.

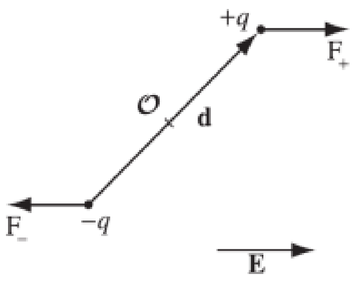
$$\mathbf{p} = \alpha \mathbf{E}.$$

The constant of proportionality α is called **atomic polarizability.**

Some molecules have built-in permanent dipole moment. When such polar molecules are placed in an electric field:

→ If the field is uniform, the force on the positive end exactly cancels the force on the negative end.

→ There is a torque.



$$\mathbf{N} = (\mathbf{r}_+ \times \mathbf{F}_+) + (\mathbf{r}_- \times \mathbf{F}_-)$$
$$= \left[(\mathbf{d}/2) \times (q\mathbf{E})\right] + \left[(-\mathbf{d}/2) \times (-q\mathbf{E})\right] = q\mathbf{d} \times \mathbf{E}.$$

Hence, we can write,

$$\mathbf{N} = \mathbf{p} \times \mathbf{E}.$$

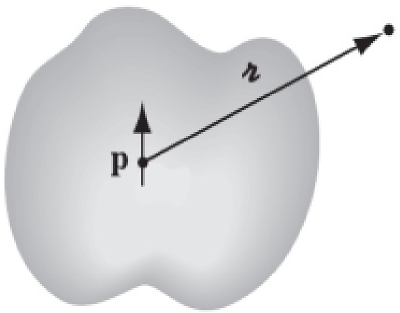
Polarization

Notice that these two mechanisms produce the same basic result: *a lot of little dipoles pointing along the direction of the field*—the material becomes **polarized**. A convenient measure of this effect is

$$\mathbf{P} \equiv \textit{dipole moment per unit volume,}$$

which is called the **polarization**.

Say, we have a polarized material with all the dipoles pointing in the same diection.



→ For a single dipole, the electrostatic potential can be written as

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$$

where, $\boldsymbol{\mathscr{r}}$ is the vector from the dipole to the point at which we are trying to find the potential. We have a dipole moment $\mathbf{p} = \mathbf{P}\,d\tau'$ in each volume element $d\tau'$. Hence the total potential is

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}} \frac{\mathbf{P}(\mathbf{r}')\cdot\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\,d\tau'$$

Field due to polarized object

The total potential can be written as,

$$V = \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}} \mathbf{P}\cdot\nabla'\left(\frac{1}{\mathscr{r}}\right) d\tau' \qquad \text{since,} \qquad \nabla'\left(\frac{1}{\mathscr{r}}\right) = \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$$

Using the identity $\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$

$$V = \frac{1}{4\pi\epsilon_0}\left[\int_{\mathcal{V}} \nabla'\cdot\left(\frac{\mathbf{P}}{\mathscr{r}}\right) d\tau' - \int_{\mathcal{V}} \frac{1}{\mathscr{r}}(\nabla'\cdot\mathbf{P})\, d\tau'\right]$$

Using divergence theorem,

$$V = \frac{1}{4\pi\epsilon_0}\oint_{\mathcal{S}} \frac{1}{\mathscr{r}}\mathbf{P}\cdot d\mathbf{a}' - \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}} \frac{1}{\mathscr{r}}(\nabla'\cdot\mathbf{P})\, d\tau'$$

Potential due to a surface charge distribution

$$\sigma_b \equiv \mathbf{P}\cdot\hat{\mathbf{n}}$$

Potential due to a volume charge distribution

$$\rho_b \equiv -\nabla\cdot\mathbf{P}$$

Bound charges

Potential of a polarized object is the same as that produced by a volume and a surface charge distribution.

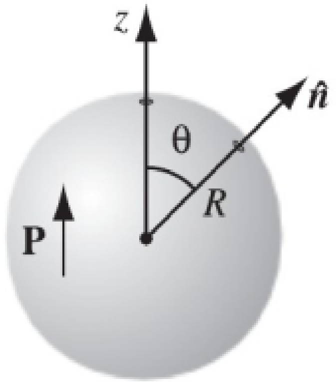
Bound charges

$$\sigma_b \equiv \mathbf{P} \cdot \hat{\mathbf{n}}$$

$$\rho_b \equiv -\nabla \cdot \mathbf{P}$$

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \oint_{\mathcal{S}} \frac{\sigma_b}{\mathcal{r}}\, da' + \frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}} \frac{\rho_b}{\mathcal{r}}\, d\tau'$$

Ex: Find the electric field produced by a uniformly polarized sphere of radius R



→ The volume bound charge density ρ_b is zero, since **P** is uniform

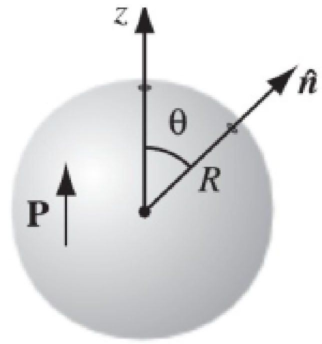
→ choose the z axis to coincide with the direction of polarization

$$\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}} = P\cos\theta$$

→ charge density $P\cos\theta$ plastered over the surface of a sphere

Bound charges

We know the expression for electric potential for such a system



$$V(r,\theta) = \begin{cases} \dfrac{P}{3\epsilon_0} r\cos\theta, & \text{for} \quad r \leq R, \\ \dfrac{P}{3\epsilon_0}\dfrac{R^3}{r^2}\cos\theta, & \text{for} \quad r \geq R. \end{cases}$$

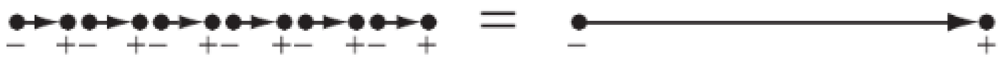
Since $r\cos\theta = z$, the *field* inside the sphere is *uniform*:

$$\mathbf{E} = -\nabla V = -\frac{P}{3\epsilon_0}\hat{\mathbf{z}} = -\frac{1}{3\epsilon_0}\mathbf{P}, \qquad \text{for} \quad r < R.$$

Bound charge: Physical interpretation

Physical interpretation of Bound Charges:

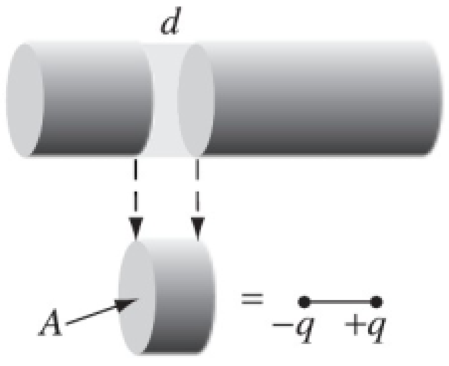
Say, we have a string of dipoles made up of same amount of +ve and -ve charges.



→ Bound Charges

To calculate the amount of bound charges resulting from a polarization

→ A tube of dielectric parallel to the polarization vector



The dipole moment of the tiny chunk shown

→ $P(Ad)$, where A is the cross-sectional area of the tube and d is the length of the chunk.

→ In terms of the charge (q) at the end, this same dipole moment can be written qd.

The bound charge that piles up at the right end of the tube is $q = PA$

Bound charge: Physical interpretation

If the ends have been sliced off perpendicularly, the surface charge density is $\sigma_b = \frac{q}{A} = P$

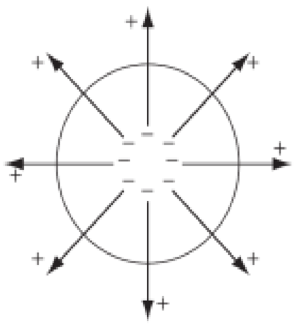
For an oblique cut the *charge* is still the same, but $A = A_{\text{end}} \cos\theta$, so

$$A = A_{\text{end}} \cos\theta$$

$$\sigma_b = \frac{q}{A_{\text{end}}} = P\cos\theta = \mathbf{P}\cdot\hat{\mathbf{n}}$$

The effect of the polarization, then, is to paint a bound charge $\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}}$ over the surface of the material.

➔ If the polarization is nonuniform, we get accumulations of bound charge *within* the material, as well as on the surface.



No net bound charge

Ex: Prove that the total bound charge is zero.

Total charge $Q = \oint_{surf} \sigma_b \, da' + \int_{vol} \rho_b \, d\tau'$

$$= \oint_{surf} \mathbf{P}(\mathbf{r}') \cdot \hat{\mathbf{n}}' da' - \int_{vol} \nabla' \cdot \mathbf{P}(\mathbf{r}') d\tau'$$

$$= \oint_{surf} \mathbf{P}(\mathbf{r}') \cdot d\mathbf{a}' - \int_{vol} \nabla' \cdot \mathbf{P}(\mathbf{r}') d\tau'$$

$$= \int_{vol} \nabla' \cdot \mathbf{P}(\mathbf{r}') d\tau' - \int_{vol} \nabla' \cdot \mathbf{P}(\mathbf{r}') d\tau' = 0$$

Gauss' law in presence of dielectrics

The net effect of Polarization is the accumulation of bound charges.

$$\sigma_b \equiv \mathbf{P} \cdot \hat{\mathbf{n}} \qquad \rho_b \equiv -\nabla \cdot \mathbf{P}$$

These bound charges give rise to electric field.

→ Hence with in a dielectric, the total electric field = *Field due to the free charges*

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Field due to bound charges

→ The total charge density, $\rho = \rho_b + \rho_f$

→ Using Gauss' law, $\epsilon_0 \nabla \cdot \mathbf{E} = \rho = \rho_b + \rho_f = -\nabla \cdot \mathbf{P} + \rho_f$

where $\mathbf{E}$ is now the *total* field, not just that portion generated by polarization.

Hence,

$$\nabla \cdot (\epsilon_0 \mathbf{E} + \mathbf{P}) = \rho_f$$