Gauss' law in presence of dielectrics

We write, $\mathbf{D} \equiv \epsilon_0 \mathbf{E} + \mathbf{P}$

→ Electric Displacement

→ In terms of **D**, Gauss's law reads

$$\nabla \cdot \mathbf{D} = \rho_f$$

Hence, the integral form:

$$\oint \mathbf{D} \cdot d\mathbf{a} = Q_{f_{enc}}$$

→ $Q_{f_{enc}}$ denotes the total free charge enclosed in the volume.

This is particularly useful since we only need to know the free charge in the system, which is something we can control. The bound charges appear only after polarization. In a typical problem, initially we just start with the free charge densities.

Linear dielectrics

Linear dielectrics is a class of dielectrics for which the induced polarization is proportional to the applied electric field.

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}.$$

χ_e → **Electric susceptibility** of the medium; depends on microscopic structure of the substance in question and also on external conditions such as temperature.

In linear media we have

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} = \epsilon_0 \mathbf{E} + \epsilon_0 \chi_e \mathbf{E} = \epsilon_0 (1 + \chi_e) \mathbf{E},$$

so **D** is *also* proportional to **E**:

$$\mathbf{D} = \epsilon \mathbf{E},$$

where

$$\epsilon \equiv \epsilon_0 (1 + \chi_e).$$

ϵ_0 → Permittivity of free space

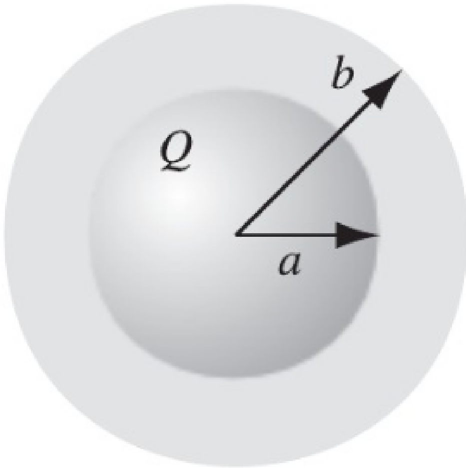
ϵ → Permittivity of a material

(Dielectric Constant)

$$\epsilon_r \equiv 1 + \chi_e = \frac{\epsilon}{\epsilon_0}$$

Linear dielectrics: Example

A metal sphere of radius a carries a charge Q. It is surrounded, out to radius b, by linear dielectric material of permittivity ϵ. Find the potential at the center (relative to infinity).



→ let's begin by calculating **D**,

$$\mathbf{D} = \frac{Q}{4\pi r^2}\hat{\mathbf{r}}, \quad \text{for all points } r > a.$$

Then,

$$\mathbf{E} = \begin{cases} \dfrac{Q}{4\pi\epsilon r^2}\hat{\mathbf{r}}, & \text{for } a < r < b, \\ \dfrac{Q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}, & \text{for } r > b. \end{cases}$$

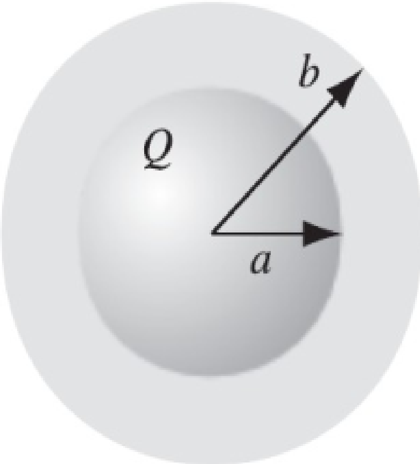
Linear dielectrics: Example

The potential at the center is therefore

$$V = -\int_{\infty}^{0} \mathbf{E}\cdot d\mathbf{l} = -\int_{\infty}^{b}\left(\frac{Q}{4\pi\epsilon_0 r^2}\right)dr - \int_{b}^{a}\left(\frac{Q}{4\pi\epsilon r^2}\right)dr - \int_{a}^{0}(0)\,dr$$

$$= \frac{Q}{4\pi}\left(\frac{1}{\epsilon_0 b} + \frac{1}{\epsilon a} - \frac{1}{\epsilon b}\right).$$

→ *To find the bound charges:*



$$\mathbf{P} = \epsilon_0\chi_e\mathbf{E} = \frac{\epsilon_0\chi_e Q}{4\pi\epsilon r^2}\hat{\mathbf{r}},$$

in the dielectric, and hence

$$\rho_b = -\nabla\cdot\mathbf{P} = 0,$$

while

$$\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} = \begin{cases} \dfrac{\epsilon_0\chi_e Q}{4\pi\epsilon b^2}, & \text{at the outer surface,} \\ \dfrac{-\epsilon_0\chi_e Q}{4\pi\epsilon a^2}, & \text{at the inner surface.} \end{cases}$$

(unit vector points outward with respect to the dielectrics)