Electric field in presence of dielectric medium

For electric displacement vector,

$$\nabla \cdot \mathbf{D} = \rho_f \quad \text{and} \quad \nabla \times \mathbf{D} = \mathbf{0}$$

→ $\mathbf{D}$ can be found from the free charge just as though the dielectric were not there:

$$\mathbf{D} = \epsilon_0 \mathbf{E}_{\text{vac}}$$

where $\mathbf{E}_{\text{vac}}$ is the field the same free charge distribution would produce in the absence of any dielectric.

→ Inside linear dielectric,

$$\mathbf{E} = \frac{1}{\epsilon}\mathbf{D} = \frac{1}{\epsilon_r}\mathbf{E}_{\text{vac}}$$

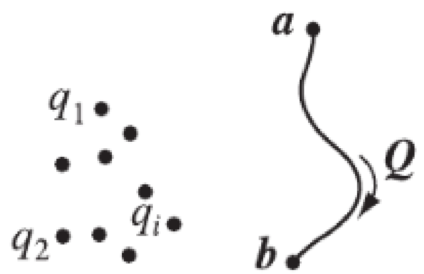
Conclusion: When all space is filled with a homogeneous linear dielectric, the field everywhere is simply reduced by a factor of one over the dielectric constant.

For example, if a free charge q is embedded in a large dielectric, the field it produces is

$$\mathbf{E} = \frac{1}{4\pi\epsilon}\frac{q}{r^2}\hat{\mathbf{r}} \quad (\text{that's } \epsilon, \text{ not } \epsilon_0)$$

Work and energy

We have a stationary charge distribution. A test charge is being moved from point **a** to point **b** in its presence,



- At any point along the path, force acting on the test charge: $\mathbf{F} = Q\mathbf{E}$
- This is the minimum amount of force one has to exert in order to move the test charge against the Coulomb force.
- Work done:

$$W = \int_{\mathbf{a}}^{\mathbf{b}} \mathbf{F} \cdot d\mathbf{l} = -Q \int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l} = Q[V(\mathbf{b}) - V(\mathbf{a})]$$

- Work done is independent of the path, only depends on the end points; ***electrostatic force is conservative***. Potential difference between point **a** and **b** is equal to the work done per unit charge.

$$V(\mathbf{b}) - V(\mathbf{a}) = \frac{W}{Q}.$$

- If I bring the test charge from infinity to some point with position vector **r**, we can write

$$W = Q[V(\mathbf{r}) - V(\infty)]$$

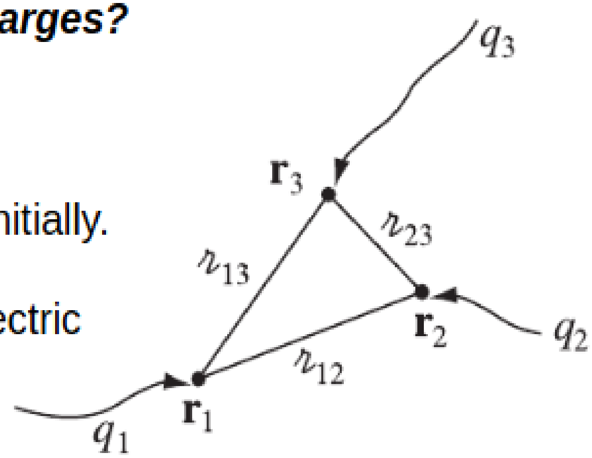
Or,

$$W = QV(\mathbf{r}).$$

Work done to assemble collection of point charges

How much work would it take to assemble an entire collection of point charges?

Imagine bringing the charges one by one from far away.



- Bringing in the first charge q_1 takes no work since there is no electric field initially.
- In order to bring in the second charge q_2 one needs to work against the electric field of the first charge.

$$W_2 = \frac{1}{4\pi\epsilon_0} q_2 \left(\frac{q_1}{\mathscr{r}_{12}} \right)$$

- Similarly, in order to bring in the next charge q_3 work done is

$$W_3 = \frac{1}{4\pi\epsilon_0} q_3 \left(\frac{q_1}{\mathscr{r}_{13}} + \frac{q_2}{\mathscr{r}_{23}} \right)$$

and for the next one,

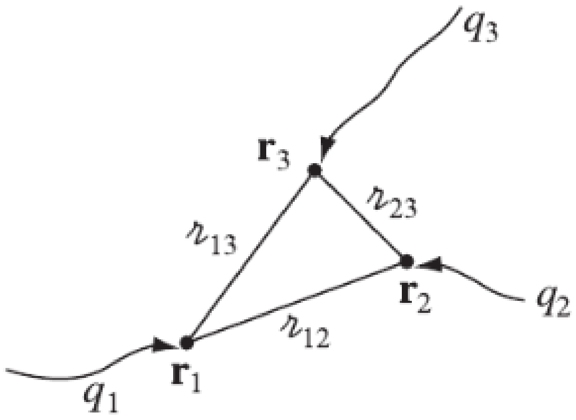
$$W_4 = \frac{1}{4\pi\epsilon_0} q_4 \left(\frac{q_1}{\mathscr{r}_{14}} + \frac{q_2}{\mathscr{r}_{24}} + \frac{q_3}{\mathscr{r}_{34}} \right)$$

and so on.....

Work done to assemble collection of point charges

Hence, total work necessary to assemble the first four charges:

$$W = \frac{1}{4\pi\epsilon_0}\left(\frac{q_1q_2}{\mathscr{r}_{12}} + \frac{q_1q_3}{\mathscr{r}_{13}} + \frac{q_1q_4}{\mathscr{r}_{14}} + \frac{q_2q_3}{\mathscr{r}_{23}} + \frac{q_2q_4}{\mathscr{r}_{24}} + \frac{q_3q_4}{\mathscr{r}_{34}}\right)$$



→ In general, for *n* number of charges one can write,

$$W = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\sum_{j>i}^{n}\frac{q_iq_j}{\mathscr{r}_{ij}}$$

The restriction on the limit of the second sum, $j > i$ is to avoid double counting.

One can also write it as:

$$W = \frac{1}{8\pi\epsilon_0}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\frac{q_iq_j}{\mathscr{r}_{ij}}$$ (also represents the energy stored in the charge configuration)

Hence,

$$W = \frac{1}{2}\sum_{i=1}^{n} q_i\left(\sum_{j\neq i}^{n}\frac{1}{4\pi\epsilon_0}\frac{q_j}{\mathscr{r}_{ij}}\right)$$ Or, $$W = \frac{1}{2}\sum_{i=1}^{n} q_i V(\mathbf{r}_i).$$

→ Potential at $\boldsymbol{r}_i$ due to all other charges.

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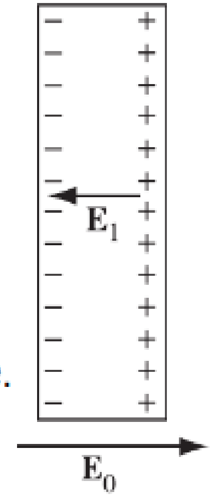
Conductors

A perfect conductor has an unlimited supply of free electrons.

→ In reality, there are no ideal conductors, but metals come pretty close.

Electric field inside a conductor is zero.



→ If a conductor is put in an external electric field, the induced charges produce an electric field inside the conductor such that the field inside is oriented opposite to the direction of the external electric field. The two fields cancel each other inside. the charges flow inside until this equilibrium is reached.

Hence, inside the conductor the charge density must be zero since

$\nabla \cdot \mathbf{E} = \rho / \epsilon_0$ → If electric field inside is zero, so is the charge density, which means there are equal number of positive and negative charges.

Any net charge it may have, can only reside on the surface. Conductors have equipotential surfaces.

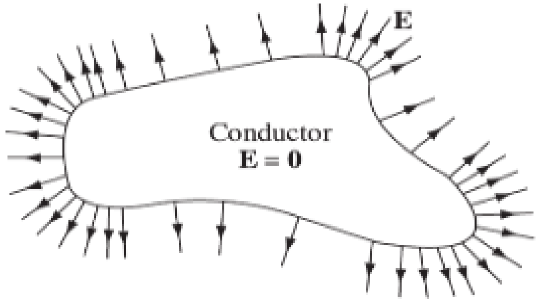
→ If **a** and **b** are two points on the surface of a conductor,

$$V(\mathbf{b}) - V(\mathbf{a}) = -\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l} = 0$$

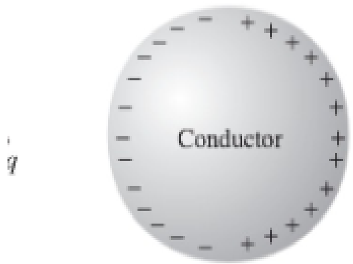
Hence, $V(\mathbf{a}) = V(\mathbf{b})$.

Electric field just outside the conductor is perpendicular to the surface.

→ Otherwise, the charges would flow around until they kill off the tangential component.

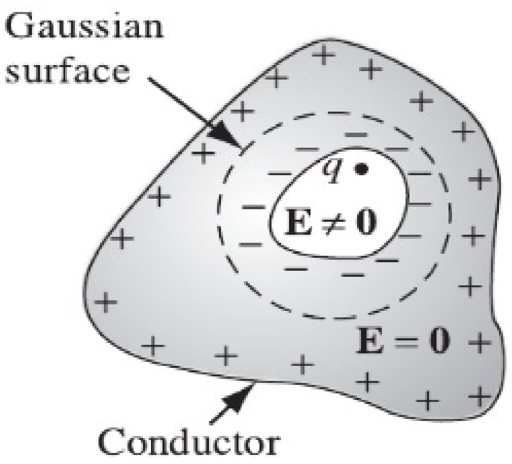


Induced Charges



If you hold a charge $+q$ near an uncharged conductor, the two will attract one another. The reason for this is that q will pull minus charges over to the near side and repel plus charges to the far side. The charges move around in such a way as to kill off the electric field inside the conductor. Since the negative induced charge is closer to q, there is a net force of attraction.

Cavity inside conductor



- Outside the conductor electric field is non-zero
- Outside the cavity but inside the conductor electric field is zero
- Inside the cavity electric field is non-zero
- The total charge induced on the cavity wall is equal and opposite to the charge inside the cavity.

For the Gaussian surface shown in the figure, $\oint \mathbf{E} \cdot d\mathbf{a} = 0$ since, we are inside the conductor but outside cavity

Here,

$$Q_{\text{enc}} = q + q_{\text{induced}}$$

Hence,

$$q_{\text{induced}} = -q.$$

- Field outside the conductor: $\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}$

Capacitors

Say, we have two conductors carrying +*Q* and -*Q* charges respectively.

Since the potential is uniform over any conductor, the potential difference between them

$$V = V_+ - V_- = -\int_{(-)}^{(+)} \mathbf{E} \cdot d\mathbf{l}$$

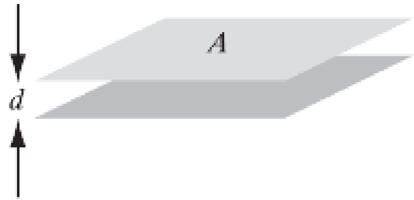
V is proportional to the electric charge, since the electric field is proportial to the charge.

⟶ The proportionality constant is called the ***capacitance(C)*** of the system.

$$C \equiv \frac{Q}{V}$$

Capacitance depends upon the sizes, shapes and distance between the two conductors.

For a parallel plate capacitor with +*Q* and -*Q* charges uniformly distributed over the plates,,



The uniform charge density, $\sigma = Q/A$

Electric field in between the plates is $(1/\epsilon_0)Q/A$

Potential difference: $V = \dfrac{Q}{A\epsilon_0}d$

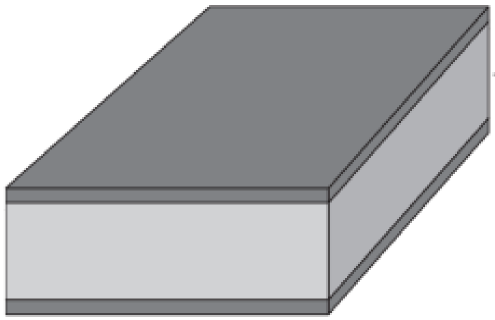
Capacitance: $C = \dfrac{A\epsilon_0}{d}.$

Dielectrics to enhance capacitance

A parallel-plate capacitor is filled with insulating material of dielectric constant ϵ_r. What effect does this have on its capacitance?

Since the field is confined to the space between the plates, the dielectric will reduce $\mathbf{E}$, and hence also the potential difference V, by a factor $1/\epsilon_r$. Accordingly, the capacitance $C = Q/V$ is *increased by a factor of the dielectric constant,*

$$C = \epsilon_r C_{\text{vac}}$$



This is, in fact, a common way to beef up a capacitor.