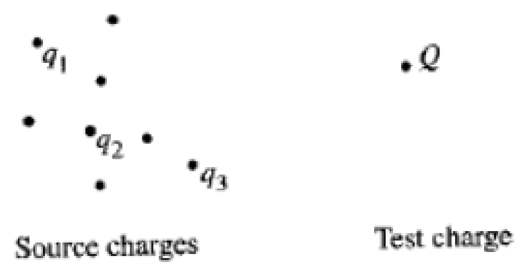
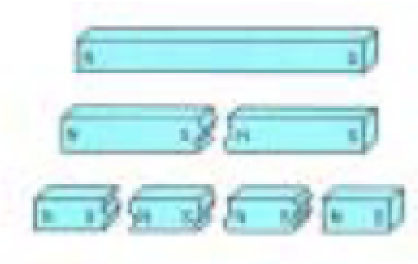
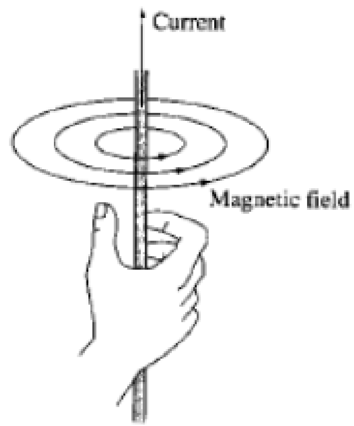
MAGNETOSTATICS

Magnets and magnetic field





If one try to isolate the poles by cutting the magnet, a curious thing happens: One obtains two magnets. No matter how thinly the magnet is sliced, each fragment always have two poles. Even down to the atomic level, no one has found an isolated magnetic pole, called a monopole. Thus magnetic field lines form closed loops.



When defining of the electric field, the electric field strength can be derived from the following relation: $\mathbf{E}=\mathbf{F}/q$. Since an isolated pole is not available, *the definition of the magnetic field is not as simple*.

How a current-carry wire produces a magnetic field?

Magnetostatics

Stationary charges ⟶ Constant Electric field ⟶ Electrostatics

Steady currents ⟶ Constant Magnetic field ⟶ Magnetostatics

Force on a point charge *Q*:

Electric Force: $\mathbf{F}_{\text{elec}} = Q\mathbf{E}$

Magnetic Force: $\mathbf{F}_{\text{mag}} = Q(\mathbf{v} \times \mathbf{B})$ **Lorentz Force Law**

Total Force: $$\boxed{\mathbf{F} = \mathbf{F}_{\text{elec}} + \mathbf{F}_{\text{mag}} = Q(\mathbf{E} + \mathbf{v} \times \mathbf{B})}$$

Work done by magnetic force

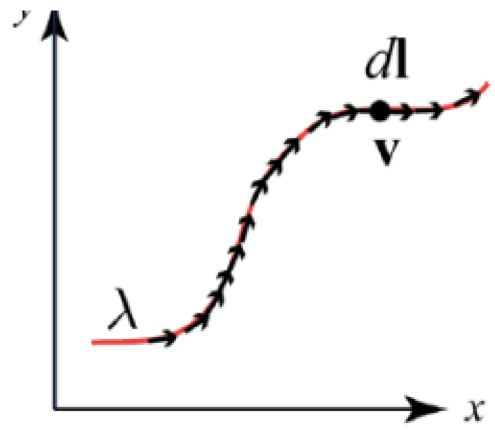
$$\mathbf{W}_{\text{mag}} = \int \mathbf{F}_{\text{mag}} \cdot d\mathbf{l} = \int Q(\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l} = \int Q(\mathbf{v} \times \mathbf{B}) \cdot \mathbf{v}\, dt$$

$$= 0$$

- Magnetic forces do no work
- Magnetic forces can change the direction in which a particle moves.
- Magnetic forces do not change the speed with which a particle moves.

Current

- Current is charge flow per unit time $I = \frac{dq}{dt}$
- It is measured in Coulombs per second, or Amperes (A).



Charge flowing in a wire is described by **Current**

$$\mathbf{I} = \frac{dq}{dt} = \frac{\lambda\, d\mathbf{l}}{dt} = \lambda \mathbf{v}$$

- The direction of current is in the direction of charge-flow.
- Conventionally, this is the direction opposite to the flow of electrons.

$$\mathbf{F}_{\text{mag}} = \int (\mathbf{I} \times \mathbf{B})\, dl = I \int (d\mathbf{l} \times \mathbf{B})$$

Magnetic force on a current carrying wire:

$$\mathbf{F}_{\text{mag}} = (\mathbf{v} \times \mathbf{B})Q = \int (\mathbf{v} \times \mathbf{B})\, dq = \int (\mathbf{v} \times \mathbf{B})\, \lambda dl = \int (\mathbf{I} \times \mathbf{B})\, dl$$

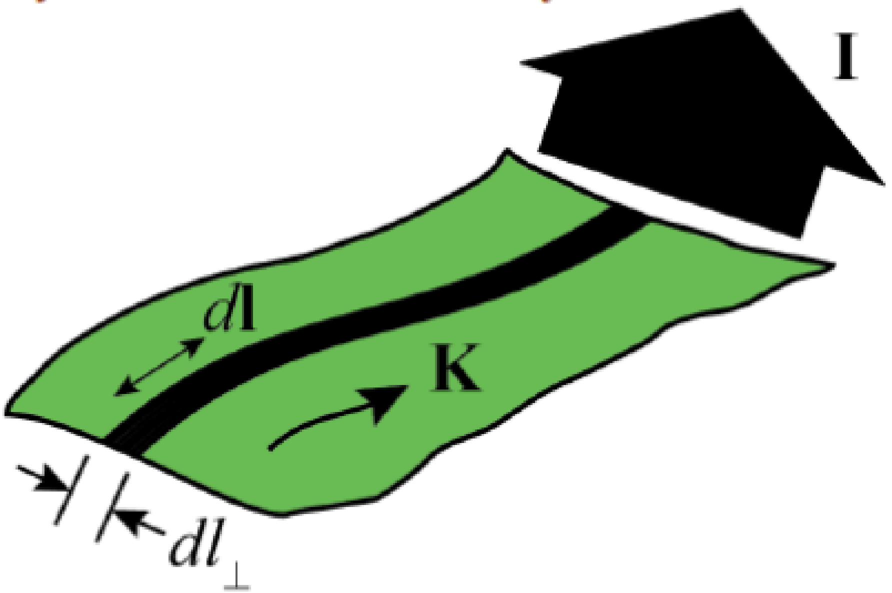
Current

- Current is charge flow per unit time $I = \frac{dq}{dt}$
- It is measured in Coulombs per second, or Amperes (A).

Charge flowing on a surface is described by **surface current density**

$$\mathbf{K} = \frac{d\mathbf{I}}{dl_\perp} = \sigma \mathbf{v}$$

- Current density is a vector quantity.



Magnetic force on the surface current:

$$\mathbf{F}_{\mathbf{mag}} = (\mathbf{v} \times \mathbf{B})Q = \int (\mathbf{v} \times \mathbf{B})\, \sigma da = \int (\mathbf{K} \times \mathbf{B})\, da$$

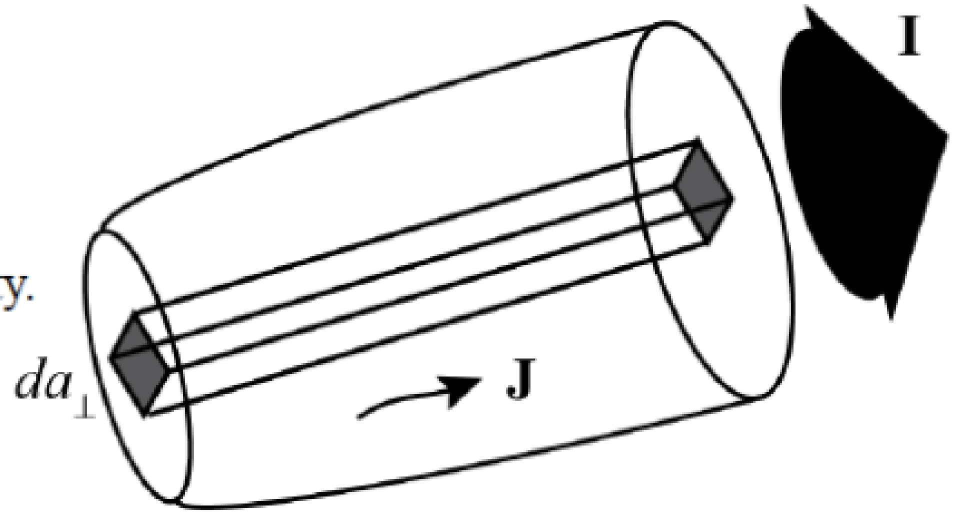
Current

- Current is charge flow per unit time $I = \frac{dq}{dt}$
- It is measured in Coulombs per second, or Amperes (A).

Charge flowing in a volume is described by **volume current density**

$$\mathbf{J} = \frac{d\mathbf{I}}{da_{\perp}} = \rho\mathbf{v}$$

- Current density is a vector quantity.



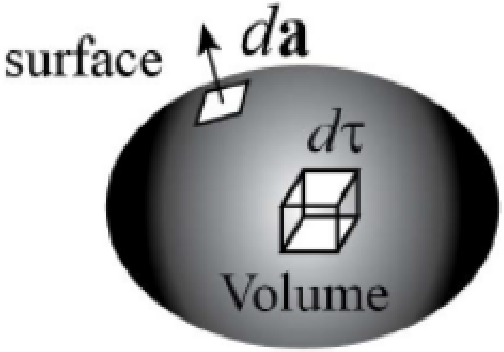
Magnetic force on the volume current:

$$\mathbf{F}_{\mathbf{mag}} = (\mathbf{v} \times \mathbf{B})Q = \int (\mathbf{v} \times \mathbf{B})\, \rho d\tau = \int (\mathbf{J} \times \mathbf{B})\, d\tau$$

Continuity equation (Conservation of charge)

$$\mathbf{J} = \frac{d\mathbf{I}}{da_\perp} \qquad \Rightarrow \mathrm{I} = \int_S \mathrm{J}\, da_\perp$$

$$\Rightarrow \mathrm{I} = \int_S \mathbf{J} \cdot d\mathbf{a}$$



For a closed surface

$$\mathrm{I} = \oint_S \mathbf{J} \cdot d\mathbf{a}$$ Total charge per unit time crossing a closed surface

$$\oint_S \mathbf{J} \cdot d\mathbf{a} = \int_V (\nabla \cdot \mathbf{J}) d\tau$$ Total charge per unit time leaving the volume V.

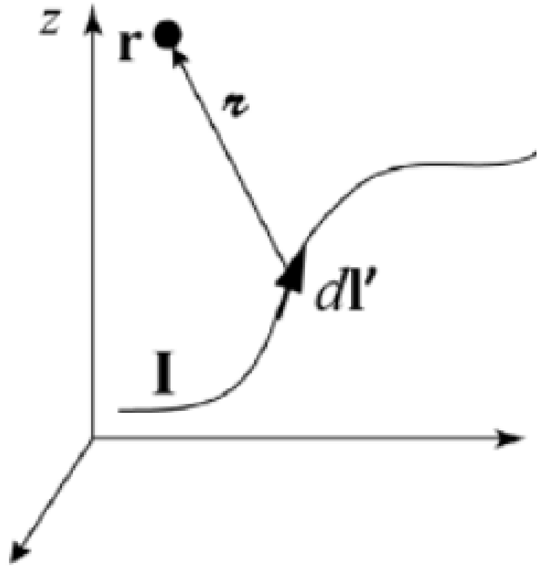
But, total charge per unit time leaving the volume V is $-\frac{d}{dt}\left(\int_V \rho\, d\tau\right) = -\int_V \frac{d\rho}{dt}\, d\tau$

So, $\int_V (\nabla \cdot \mathbf{J}) d\tau = -\int_V \frac{d\rho}{dt}\, d\tau \Rightarrow$ $$\boxed{\nabla \cdot \mathbf{J} = -\frac{d\rho}{dt}}$$ **The Continuity Equation**

Biot-Savart law

The magnetic field produced by a steady line current

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{I}\times\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \qquad = \frac{\mu_0}{4\pi} I \int \frac{d\mathbf{l}'\times\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$$

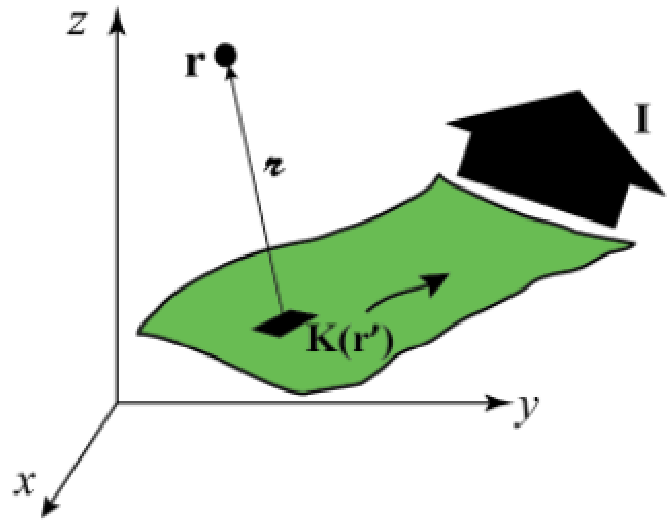


- μ_0 is the permeability of free space
- $\mu_0 = 4\pi \times 10^{-7}\ \mathrm{N/A^2}$
- The unit of magnetic field is Newton per Ampere-meter, or Tesla
- 1 Tesla is a very strong magnetic field. Earth's magnetic field is about 10^{-4} times smaller
- Biot-Savart law for magnetic field is analogous to Coulomb's law for electric field

Biot-Savart law

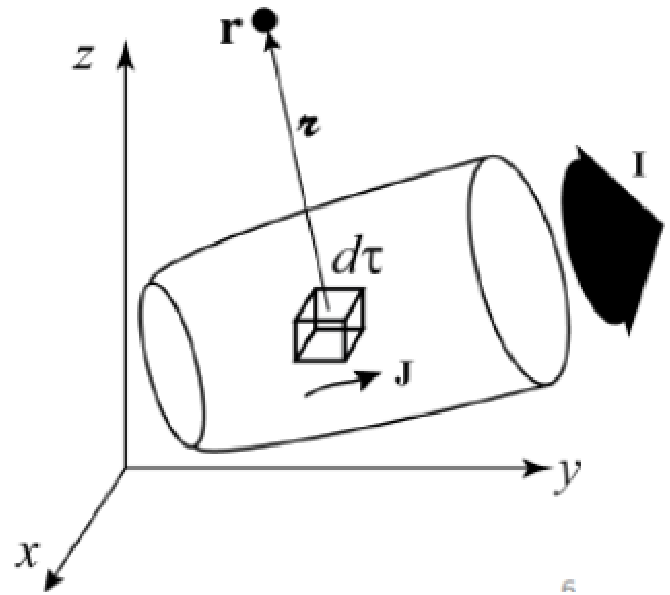
The magnetic field produced by a surface current

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{K}(\mathbf{r}') \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\, da'$$



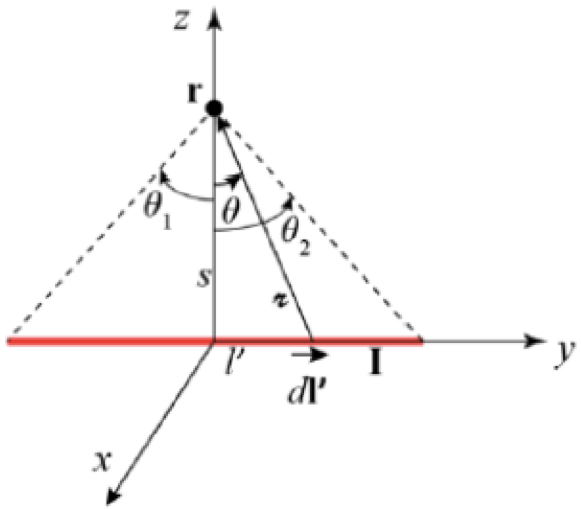
The magnetic field produced by a volume current

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}') \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\, d\tau'$$



Biot-Savart law: Example

Ex. 5.5 (Griffiths, 3rd Ed.): Calculate the magnetic field due to a long straight wire carrying a steady current I .



$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} I \int \frac{d\mathbf{l}' \times \hat{\boldsymbol{\imath}}}{\imath^2}$$

$$\mathbf{B}(\mathbf{r}) = B\,\hat{\mathbf{x}}$$

$$|d\mathbf{l}' \times \hat{\boldsymbol{\imath}}| = dl' \cos\theta$$

$$l' = s\tan\theta \quad \Rightarrow \quad dl' = \frac{s}{\cos^2\theta} d\theta$$

$$s = \imath\cos\theta \quad \Rightarrow \quad \frac{1}{\imath^2} = \frac{\cos^2\theta}{s^2}$$

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} I \int_{\theta_1}^{\theta_2} \left(\frac{\cos^2\theta}{s^2}\right)\left(\frac{s}{\cos^2\theta}\right) \cos\theta \, d\theta \, \hat{\mathbf{x}}$$

$$= \frac{\mu_0 I}{4\pi s} \int_{-\theta_1}^{\theta_2} \cos\theta \, d\theta \, \hat{\mathbf{x}}$$

$$= \frac{\mu_0 I}{4\pi s} (\sin\theta_2 + \sin\theta_1)\hat{\mathbf{x}}$$

Biot-Savart law: Example

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} I \int \frac{d\mathbf{l}' \times \hat{\boldsymbol{\imath}}}{\imath^2}$$

$$= \frac{\mu_0 I}{4\pi s} (\sin\theta_2 + \sin\theta_1)\hat{\mathbf{x}}$$

Field due to an infinite wire ?

$$\theta_1 = \frac{\pi}{2} \qquad \theta_2 = \frac{\pi}{2}$$

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi s} (\sin\theta_2 + \sin\theta_1)\hat{\mathbf{x}}$$

$$= \frac{\mu_0 I}{4\pi s} (1 + 1)\hat{\mathbf{x}}$$

$$= \frac{\mu_0 I}{2\pi s}\hat{\mathbf{x}}$$