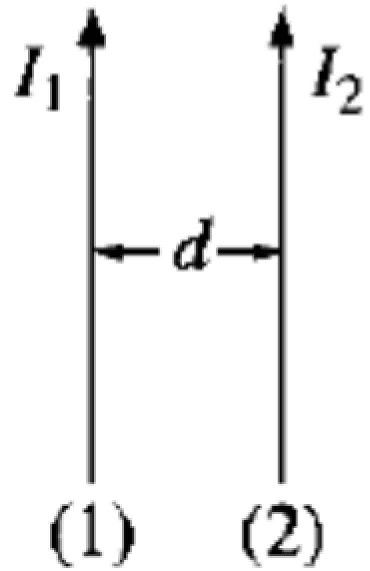
Force between two current carrying wires

$$d\mathbf{F} = I dl \times \mathbf{B}$$

$$dF = I_2 \frac{\mu_0 I_1}{2\pi d} dl = \frac{\mu_0 I_1 I_2}{2\pi d} dl$$

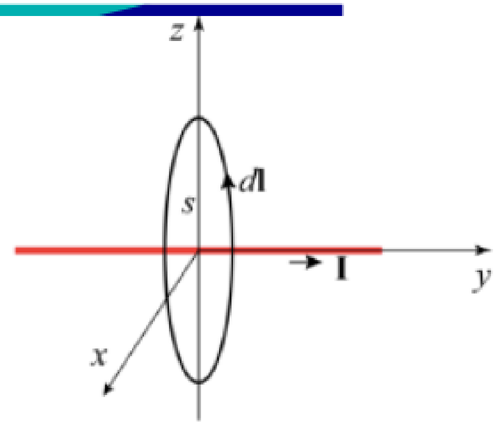
$$\frac{dF}{dl} = \frac{\mu_0 I_1 I_2}{2\pi d}$$



Divergence and Curl of B

What is the divergence of **B** ? $\nabla \cdot \mathbf{B} = 0$

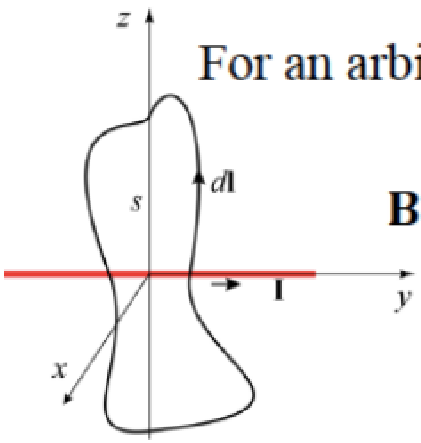
What is the curl of **B** ? Should be $\nabla \times \mathbf{B} \neq \mathbf{0}$



$$\oint \mathbf{B} \cdot d\mathbf{l} = \oint \frac{\mu_0 I}{2\pi s} dl = \frac{\mu_0 I}{2\pi s} \oint dl = \frac{\mu_0 I}{2\pi s} 2\pi s = \mu_0 I$$

The line integral is independent of *s*

$$\mathrm{B}(s) = \frac{\mu_0 I}{2\pi s}$$



For an arbitrary path enclosing the current carrying wire

cylindrical coordinate

$$\mathbf{B}(s) = \frac{\mu_0 I}{2\pi s} \hat{\boldsymbol{\phi}} \qquad d\mathbf{l} = ds\, \hat{\mathbf{s}} + s d\phi\, \hat{\boldsymbol{\phi}} + dz\, \hat{\mathbf{z}}$$

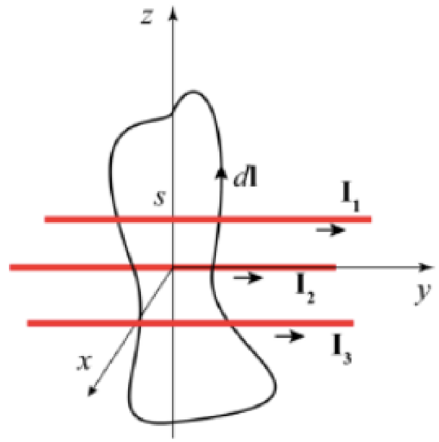
$$\oint \mathbf{B} \cdot d\mathbf{l} = \oint \frac{\mu_0 I}{2\pi s} \hat{\boldsymbol{\phi}} \cdot (ds\, \hat{\mathbf{s}} + s d\phi\, \hat{\boldsymbol{\phi}} + dz\, \hat{\mathbf{z}}) = \frac{\mu_0 I}{2\pi} \int_0^{2\pi} d\phi = \mu_0 I$$

Curl of B

If the path encloses more than one current carrying wire

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_1 + \mu_0 I_2 + \mu_0 I_3 = \mu_0 I_{\text{enc}}$$

$$I_{\text{enc}} = \int \mathbf{J} \cdot d\mathbf{a}$$



$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int \mathbf{J} \cdot d\mathbf{a}$$

$$\int (\nabla \times \mathbf{B}) \cdot d\mathbf{a} = \mu_0 \int \mathbf{J} \cdot d\mathbf{a} \quad \Rightarrow \quad \boxed{\nabla \times \mathbf{B} = \mu_0 \mathbf{J}}$$

It is valid in general

Ampere's law

$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$ Ampere's law in differential form

$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\mathrm{enc}}$ Ampere's law in integral form

- Ampere's law is analogous to Gauss's law
- Ampere's law makes the calculation of magnetic field very easy if there is symmetry.
- If there is no symmetry, one has to use Biot-Savart law to calculate the magnetic field.

Electrostatics vs Magnetostatics

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\,\rho(\mathbf{r}')d\tau$$

Coulomb's Law

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}') \times \hat{\mathscr{r}}}{\mathscr{r}^2}d\tau'$$

Biot-Savart Law

$\mathbf{F}_{\mathbf{elec}} = \mathrm{Q}\mathbf{E}$ Electric Force

$\mathbf{F}_{\mathbf{mag}} = \mathrm{Q}(\mathbf{v} \times \mathbf{B})$ Magnetic Force

$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$ Gauss's Law

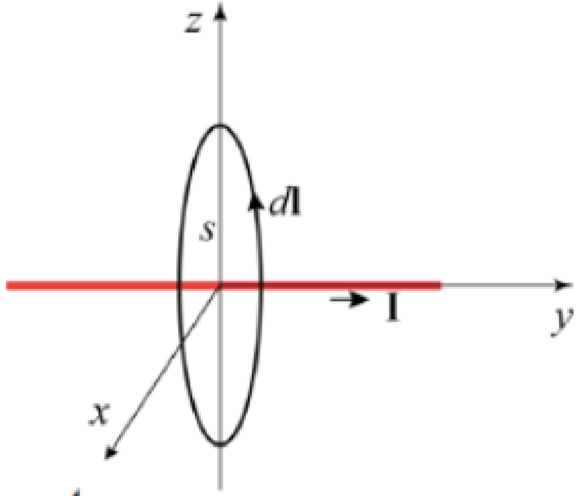
$\nabla \cdot \mathbf{B} = 0$ No Name

$\nabla \times \mathbf{E} = 0$ No Name

$\nabla \times \mathbf{B} = \mu_0\mathbf{J}$ Amperes's Law

Ampere's law: Example

Ex. 5.5 (Griffiths, 3rd Ed.): Calculate the magnetic field due to an infinitely long straight wire carrying a steady current I .



Using Biot-Savart Law

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{2\pi s}\hat{\mathbf{x}}$$

Make an Amperian loop of radius s enclosing the current

Since it is an infinite wire, the magnetic field must be circularly symmetric. Therefore,

$$\oint \mathbf{B} \cdot d\mathbf{l} = B \oint dl = B 2\pi s = \mu_0 I_{\mathbf{enc}} = \mu_0 I$$

$$B = \frac{\mu_0 I}{2\pi s}$$

Vector potential

If the divergence of a vector field $\mathbf{F}$ is zero everywhere, $(\nabla \cdot \mathbf{F} = 0)$, then:

(1) $\int \mathbf{F} \cdot d\mathbf{a}$ is independent of surface.

(2) $\oint \mathbf{F} \cdot d\mathbf{a} = 0$ for any closed surface.

This is because of the divergence theorem

$$\int_{Vol} (\nabla \cdot \mathbf{F}) d\tau = \oint_{Surf} \mathbf{F} \cdot d\mathbf{a}$$

$\nabla \cdot (\nabla \times \mathbf{A}) = 0$ $\mathbf{F}$ is the curl of a vector function: $\mathbf{F} = \nabla \times \mathbf{A}$

Magnetic Vector Potential

The vector potential is not unique. A gradient ∇V of a scalar function can be added to $\mathbf{A}$ without affecting the curl, since the curl of a gradient is zero.

$$\nabla \cdot \mathbf{B} = 0 \quad \Rightarrow \quad \mathbf{B} = \nabla \times \mathbf{A}$$