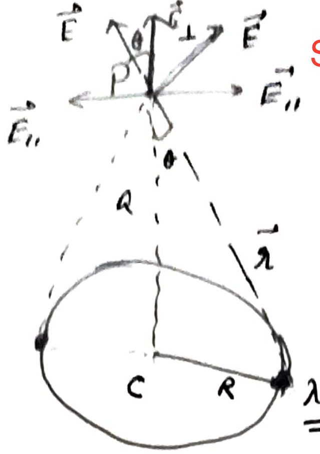
Solution Set I

Q1



$\lambda\, dl$ (λ = line-charge density $= \dfrac{Q}{2\pi R}$)

'Horizontal' components cancel each other.

Therefore the electric field at P is just the vertical component ($\cos\theta$ component):

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \int \frac{\lambda\, dl}{r^2} \cos\theta \; \hat{z}$$

($\hat{z}$ = unit vector along the vertical direction).

$$\left.\begin{aligned} r^2 &= a^2 + R^2 \\ \cos\theta &= \frac{a}{r} \end{aligned}\right] \quad \text{and} \quad \lambda = \frac{Q}{2\pi R}$$

Hence
$$\vec{E} = \frac{\lambda}{4\pi\epsilon_0} \int \frac{dl}{a^2+R^2} \cdot \frac{a}{\sqrt{a^2+R^2}} \; \hat{z}$$

$$= \frac{\lambda}{4\pi\epsilon_0} \cdot \frac{a}{(a^2+R^2)^{3/2}} \int dl \; \hat{z}$$

$$= \frac{\lambda \cdot 2\pi R}{4\pi\epsilon_0} \cdot \frac{a}{(a^2+R^2)^{3/2}} \; \hat{z}$$

$$= \frac{Q}{4\pi\epsilon_0} \cdot \frac{a}{(a^2+R^2)^{3/2}} \; \hat{z}$$

Break it into the rings of radius 'r' and thickness dr, and use the previous problem's results.

Q2

Total charge of a ring $= \sigma . 2\pi r \, dr$

(σ = surface charge density of circular disk $= \frac{Q}{\pi R^2}$)

Electric field at a distance 'z' due to the ring

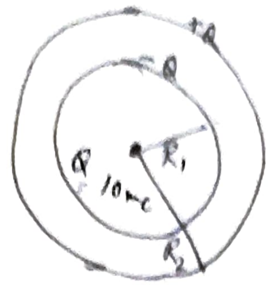
$$= \frac{\sigma . 2\pi r \, dr}{4\pi \epsilon_0} \cdot \frac{z}{(r^2+z^2)^{3/2}} \hat{z}$$

Therefore electric field for the disk at a distance z

$$\vec{E} = \frac{2\pi\sigma z}{4\pi\epsilon_0} \cdot \int_0^R \frac{r \, dr}{(r^2+z^2)^{3/2}} \hat{z}$$

$$= \frac{2\pi z}{4\pi\epsilon_0} \cdot \frac{Q}{\pi R^2} \left[\frac{1}{z} - \frac{1}{\sqrt{R^2+z^2}} \right] \hat{z}$$

$$= \frac{Qz}{2\pi\epsilon_0 R^2} \left[\frac{1}{z} - \frac{1}{\sqrt{R^2+z^2}} \right] \hat{z}$$

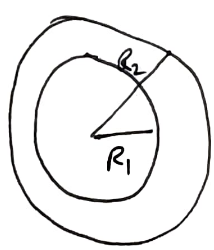


Q3

a) Total charges induced ~~the~~ at the inner surface $= -Q = -10\,mC$

Total charges induced at the outer surface $= +Q = +10\,mC$

b) Uniform



vol. charge density $= \rho$.

Q4

Three regions —

Region 1 $\rightarrow$ $0 \leq r \leq R_1$

Region 2 $\rightarrow$ $R_1 < r \leq R_2$

Region 3 $\rightarrow$ $r > R_2$

r = distance from the centre.

There is no charge in Region-1,

Hence electric field $\vec{E}_1 = 0$ $\quad 0 \leq r \leq R_1$

$\nabla \cdot \vec{E}_1 = 0$

For region - 2 ($R_1 \le r \le R_2$)

$$\oint \vec{E} \cdot \vec{ds} = \frac{Q_{enc}}{\epsilon_0}$$

$$Q_{enc} = \rho \cdot \frac{4}{3}\pi \left(r^3 - R_1^3\right)$$

$$\text{Hence } \vec{E_2} = \frac{4\pi\rho}{3\epsilon_0} \cdot \frac{1}{4\pi r^2}\left(r^3 - R_1^3\right)\hat{r}$$

$$= \left(\frac{\rho r}{3\epsilon_0} - \frac{\rho R_1^3}{3\epsilon_0 r^2}\right)\hat{r}$$

$$\nabla \cdot \vec{E_2} = \rho/\epsilon_0$$

For region - 3 ($r > R_2$)

$$Q_{enc} = \rho \cdot \frac{4\pi}{3}\left(R_2^3 - R_1^3\right)$$

$$\vec{E_3} = \frac{4\pi\rho}{3\epsilon_0} \cdot \frac{1}{4\pi r^2}\left(R_2^3 - R_1^3\right)$$

$$= \frac{\rho}{3\epsilon_0 r^2}\left(R_2^3 - R_1^3\right)\hat{r}$$

$$\nabla \cdot \vec{E_3} = 0$$

Q5

$$\rho(r) = \rho_0 + \alpha r \qquad 0 < r < R$$
$$= 0 \qquad r > R$$

a) $$Q_{enc} = \int \rho(r)\, d\tau$$

$d\tau$ in spherical polar coordinate

$$= r^2 \sin\theta \, d\theta \, d\phi \, dr$$

Total charge inside the sphere of radius R

$$Q_{en} = \int (\rho_0 + \alpha r)\, r^2 \sin\theta \, d\theta \, d\phi \, dr$$

$$= \rho_0 \int_0^R r^2 \, dr \int_0^\pi \sin\theta \, d\theta \int_0^{2\pi} d\phi$$

$$+ \alpha \int_0^R r^3 \, dr \int_0^\pi \sin\theta \, d\theta \int_0^{2\pi} d\phi$$

$$= \frac{4\pi\rho_0}{\cancel{3}} R^3/3 + \frac{4\pi\alpha}{\cancel{3}} \frac{R^4}{4}$$

$$= \frac{4\pi R^3}{3}\left(\rho_0 + \frac{3\alpha}{4} R\right)$$

b) <u>For $0 < r < R$</u>

$$Q_{enc} = \frac{4\pi r^3}{3}\left(\rho_0 + \frac{3\alpha r}{4}\right)$$

Hence $$\vec{E} = \frac{1}{4\pi r^2} \cdot \frac{4\pi r^3}{3\epsilon_0}\left(\rho_0 + \frac{3\alpha r}{4}\right)$$

$$= \frac{r}{3\epsilon_0}\left(\rho_0 + \frac{3\alpha r}{4}\right)$$

For $r > R$

$$\vec{E} = \frac{1}{4\pi r^2} \frac{4\pi R^3}{3\epsilon_0}\left(\rho_0 + \frac{3\alpha R}{4}\right)$$

$$= \frac{R^3}{3\epsilon_0 r^2}\left(\rho_0 + \frac{3\alpha R}{4}\right)$$

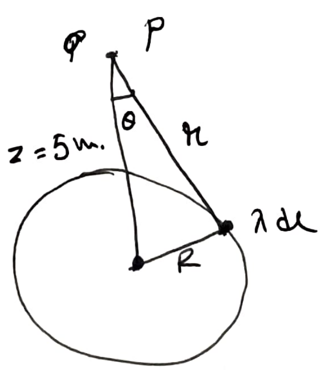
c) $\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$ for $0 < r < R$

$\vec{\nabla} \cdot \vec{E} = 0$ for $r > R$

d) $\vec{\nabla} \times \vec{E} = 0$ everywhere.

Q6

a)



radius of ring $(R) = 2$ m

λ = line charge density

$$= \frac{50 \times 10^{-9}}{2\pi \times 2} \text{ Coulomb/m}$$

$$dV(r) = \frac{\lambda \, dl}{4\pi \epsilon_0 r}$$

$$r = \sqrt{R^2 + z^2} = \sqrt{2^2 + 5^2} = \sqrt{29}$$

$$\Rightarrow V = \frac{\lambda}{4\pi\epsilon_0\sqrt{29}} \int dl$$

$$= \frac{2\pi\lambda \cdot R}{4\pi\epsilon_0\sqrt{29}}$$

$$= \frac{2\pi\lambda \cdot 2}{4\pi\epsilon_0\sqrt{29}}$$

b) Find the potential at $z = 0$ (V_1)
and at $z = 5\,m$ (V_2)

Work done $= Q(V_2 - V_1)$, here $Q = 10\,nC$.

c) potential at $z = 5$ and $z = -5$ is same.

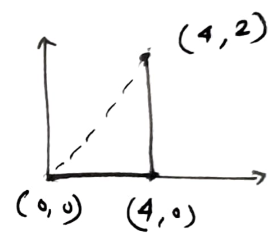
Hence in this case $V_1 = V_2$

work done $= 0$.

Q7

$$\vec{E} = 2(x+4y)\hat{i} + 8x\hat{j}$$

$$\vec{dl} = dx\hat{i} + dy\hat{j} + dz\hat{k}$$



$$\vec{E}\cdot\vec{dl} = 2(x+4y)\,dx + 8x\,dy$$

potential difference $= \int \vec{E}\cdot\vec{dl}$

$$= \left[\int_0^4 2(x+4y)\,dx\right]_{y=0} + \left[\int_0^2 8x\,dy\right]_{x=4}$$

$$= x^2\Big|_0^4 + 8xy\Big|_{y=0}^{2}\ _{x=4}$$

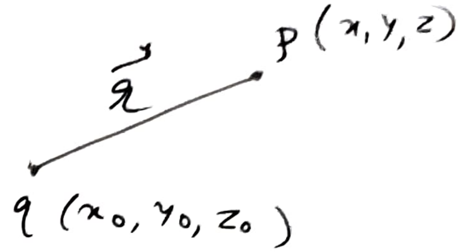
$$= (16 + 64)\ V$$

$$= 80\ V.$$

10) Q8 ~~potential due to a point charge at (x_0~~

potential at (x, y, z) due to a point charge at (x_0, y_0, z_0) can be expressed as

$$V(x,y,z) = \frac{q}{4\pi\epsilon_0 |\vec{r}|}$$



$$\vec{r} = (x - x_0)\hat{i} + (y - y_0)\hat{j} + (z - z_0)\hat{k}$$

$$|\vec{r}| = \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}$$

Calculate the potential at $(2, 2, 3)$ where

$$|\vec{r}_1| = \sqrt{(2-2)^2 + (2-3)^2 + (3-3)^2}$$

$$= 1$$

$$V_1 = \frac{1.2 \times 10^{-9}}{4\pi\epsilon_0 \times 1}$$

Similarly $|\vec{r}|$ for $(-2, 3, 3)$ is

$$|\vec{r}_2| = \sqrt{(-2-2)^2 + (3-3)^2 + (3-3)^2}$$

$$= 4$$

$$V_2 = \frac{1.2 \times 10^{-9}}{4\pi\epsilon_0 \times 4}.$$

potential difference $= (V_1 - V_2)$.