TUTORIAL SET-5

① i) $\oint_S \vec{E}\cdot d\vec{a} = \dfrac{q_{enc.}}{\epsilon_0} = \dfrac{q_2}{\epsilon_0}$

$\oint_S \vec{D}\cdot d\vec{a} = q_{f\,enc.} = q_2$

ii) $\oint_C \vec{E}\cdot d\vec{l} = 0$ (closed line integral of $\vec{E}$)

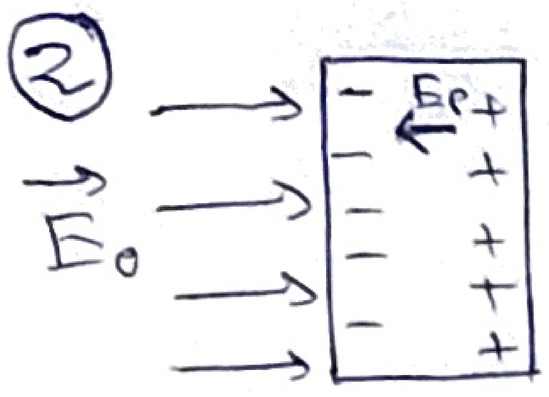
iii) $\vec{\nabla}\cdot\vec{D}\Big|_{\text{at point A}}$ (inside dielectric sphere)

$= 0$ (since $\rho_f = 0$)

iv) $\vec{\nabla}\cdot\vec{E}\Big|_{\text{at point B}}$

$= 0$

②



All the little dipoles inside will point along the direction of the field.

$\vec{E} \equiv \vec{E}_{inside} = \dfrac{\vec{E}_0}{k}$ | $k \to$ Dielectric const. (Relative Permittivity)

The induced charge will result in the electric field $\vec{E}_p$ inside the dielectric.

$\Rightarrow$ Net electric field inside:

$$\vec{E} = \frac{\vec{E_0}}{k} = \vec{E_0} - \vec{E}_p$$

$$\& \quad \frac{|\vec{E_0}|}{|\vec{E}|} = k$$

$$\Rightarrow |\vec{E}_p| = \left(1 - \frac{1}{k}\right)|\vec{E}_0|$$

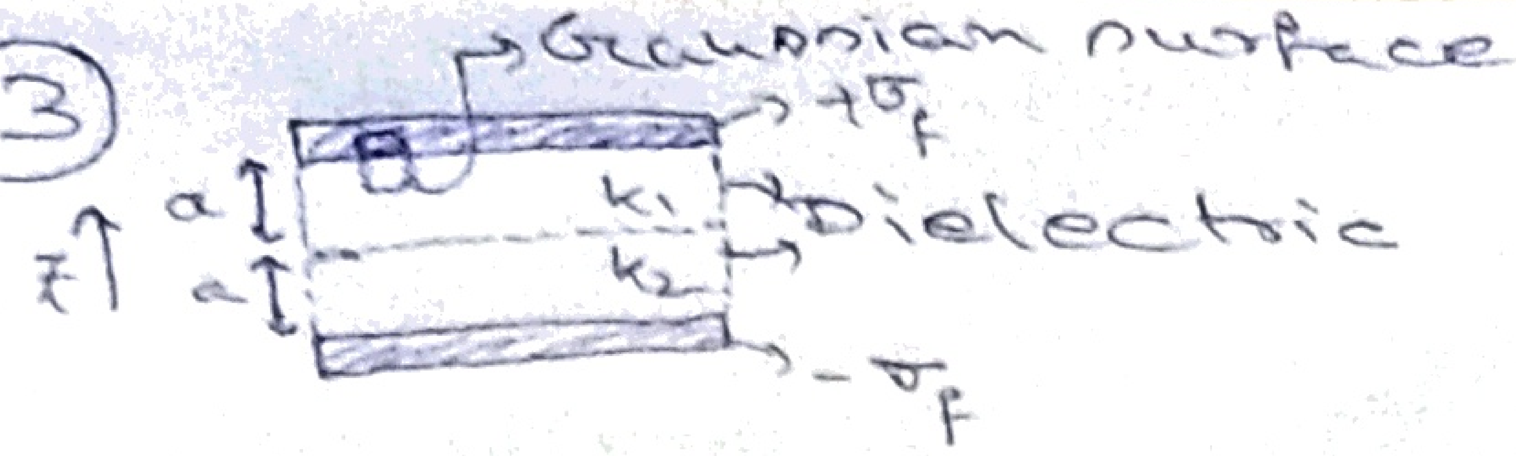
$$\Rightarrow \frac{\sigma_b}{\epsilon_0} = \frac{k-1}{k}E_0 \qquad | \ \sigma_b \equiv \text{bound surface charge}$$

$$\Rightarrow \sigma_b = \epsilon_0 \frac{k-1}{k} E_0$$

Polarisation uniform

$\Rightarrow \rho_b = 0$

3



Dielectric constant
$\epsilon_r = k_1$ for slab 1
$= k_2$ for slab 2

* $\vec{D}$ in each slab:

$$\oint \vec{D} \cdot d\vec{a} = Q_{f\,enc}.$$

↳ integration over the Gaussian surface shown in figure with upper plate & Dielectric slab with const. k_1

$$\Rightarrow DA = \sigma_f A$$

$$\Rightarrow D = \sigma_f$$

→ with the z-axis chosen as in figure, D points towards $-\hat{z}$ (upper plate has $+\sigma_f$)

$$\Rightarrow \vec{D} = \sigma_f(-\hat{z})$$ in each slab. ($\vec{D}=0$ inside metal plate)

* $\vec{P}$ in each slab:

$$\vec{D} = \epsilon \vec{E} \Rightarrow |\vec{E}_1| = \frac{\sigma_f}{\epsilon_1} \text{ in slab 1}$$

$$|\vec{E}_2| = \frac{\sigma_f}{\epsilon_2} \text{ in slab 2}$$

$$\epsilon = \epsilon_0 \epsilon_r$$

$$\Rightarrow \epsilon_1 = k_1 \epsilon_0$$
$$\epsilon_2 = k_2 \epsilon_0$$

$$\Rightarrow \vec{E}_1 = \frac{\sigma_f}{k_1 \epsilon_0}(-\hat{z})$$
$$\vec{E}_2 = \frac{\sigma_f}{k_2 \epsilon_0}(-\hat{z})$$

$$\vec{P} = \epsilon_0 \chi_e \vec{E}$$

$$\Rightarrow |\vec{P}| = \frac{\epsilon_0 \chi_e \sigma_f}{\epsilon_0 \epsilon_r} = \frac{\chi_e}{\epsilon_r}\sigma_f$$

$$= \frac{\epsilon_r - 1}{\epsilon_r}\sigma_f \qquad \Big|\, \chi_e = \epsilon_r - 1$$

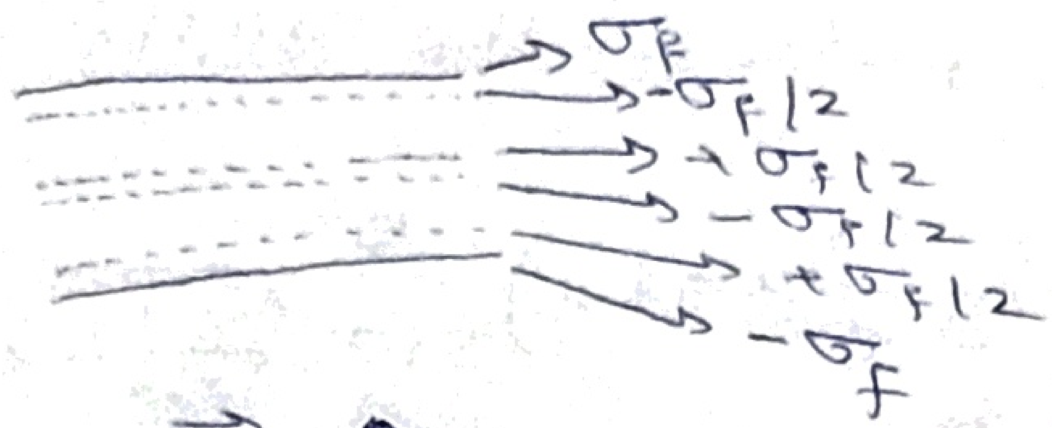
$$\Rightarrow P = \left(1 - \frac{1}{\epsilon_r}\right)\sigma_f$$

$$\Rightarrow \vec{P}_1 = \left(1 - \frac{1}{k_1}\right)\sigma_f(-\hat{z})$$

$$\vec{P}_2 = \left(1 - \frac{1}{k_2}\right)\sigma_f(-\hat{z})$$

(x) Bound charges:

(a) constant polarisation $\Rightarrow \rho_b = 0$



(x) slab ①

Normal unit vector ($\hat{n}$)

$\hat{n} = -\hat{z}$ (bottom surface)
$= +\hat{z}$ (top surface)

$\sigma_b = \vec{P}\cdot\hat{n}$

$= +P_1$ (bottom, slab 1)

$= -P_1$ (top, slab 1)

$= +P_2$ (bottom, slab 2)

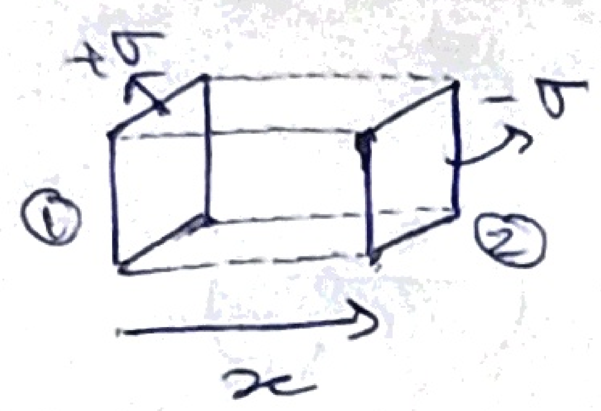
$= -P_2$ (top, slab 2)

(x) slab ②

Normal unit vector ($\hat{n}$)

$\hat{n} = -\hat{z}$ (bottom surface)
$= +\hat{z}$ (top surface)

(c)



Separation betn plates $= d$

$$\epsilon = \epsilon_1 + \frac{\epsilon_2 - \epsilon_1}{d}x$$

(c) Bound volume charge density within dielectric: $\rho_b = -\vec{\nabla}\cdot\vec{P}$

Electric displacement, $\vec{D} = \sigma\hat{x}$

Electric field, $\vec{E} = \frac{\vec{D}}{\epsilon}$

$$= \frac{\sigma}{\epsilon_1 + \frac{\epsilon_2-\epsilon_1}{d}x}\hat{x}$$

Polarisation, $\vec{P} = \epsilon_0 \chi_e \vec{E}$

$$= \epsilon_0\left(\frac{\epsilon}{\epsilon_0} - 1\right)\vec{E}$$

$$= (\epsilon - \epsilon_0)\vec{E}$$

$$= \left(\epsilon_1 + \frac{\epsilon_2-\epsilon_1}{d}x - \epsilon_0\right)\frac{\sigma}{\epsilon_1 + \frac{\epsilon_2-\epsilon_1}{d}x}\hat{x}$$

Hence,

$$\rho_b = -\vec{\nabla}\cdot\vec{P}$$

$$= -\vec{\nabla}\cdot\left[\frac{\sigma\left(\epsilon_1 + \frac{\epsilon_2-\epsilon_1}{d}x - \epsilon_0\right)}{\epsilon_1 + \frac{\epsilon_2-\epsilon_1}{d}x}\hat{x}\right]$$

$$= -\left[\frac{\sigma\left(\frac{\epsilon_2-\epsilon_1}{d}\right)}{\epsilon_1 + \frac{\epsilon_2-\epsilon_1}{d}x} - \frac{\sigma\left(\epsilon_1 + \frac{\epsilon_2-\epsilon_1}{d}x - \epsilon_0\right)}{\left(\epsilon_1 + \frac{\epsilon_2-\epsilon_1}{d}x\right)^2}\cdot\frac{(\epsilon_2-\epsilon_1)}{d}\right]$$

$$= -\frac{\sigma\epsilon_0\left(\frac{\epsilon_2-\epsilon_1}{d}\right)}{\left(\epsilon_1 + \frac{\epsilon_2-\epsilon_1}{d}x\right)^2}$$

ⓑ Bound surface charge density

$$\sigma_b = \vec{P}\cdot\hat{n}$$

$$= \vec{P}\cdot\hat{x} \quad [\text{For the plate on the right}]$$

$$= P = \left(\epsilon_1 + \frac{\epsilon_2-\epsilon_1}{d}x - \epsilon_0\right)\frac{\sigma}{\epsilon_1 + \frac{\epsilon_2-\epsilon_1}{d}x}$$

* For the plate on the left: $\hat{n} = -\hat{x}$

ⓒ Electric field, $\vec{E} = \dfrac{\sigma}{\epsilon_1 + \frac{\epsilon_2-\epsilon_1}{d}x}\hat{x}$

ⓓ Potential difference $V = -\displaystyle\int_d^0 \vec{E}\cdot d\vec{l}$

(when plate ① is ~~at~~ $x=0$)

$$= +\sigma\int_0^d \frac{dx}{\epsilon_1 + \frac{\epsilon_2-\epsilon_1}{d}x}$$

$$= \frac{\sigma d}{\epsilon_2-\epsilon_1}\ln\left(\frac{\epsilon_2}{\epsilon_1}\right)$$

Q4

Using results from problem ③

$$|\vec{D}_1| = |\vec{D}_2| = D = \sigma_f = 30\ \mu C/m^2$$

$$|\vec{E}_1| = \frac{30\times10^{-6}}{2\times8.85\times10^{-12}}\ \frac{N}{C}$$

$$\simeq 1695\times10^3\ \frac{N}{C}$$

$$= 1695\times10^3\ \frac{V}{m}$$

$$|\vec{E}_2| = \frac{30\times10^{-6}}{3\times8.85\times10^{-12}}\ \frac{V}{m}$$

$$\simeq 1130\times10^3\ \frac{V}{m}$$

Dielectric sheet ①

$$\sigma_b = \left(1-\frac{1}{\epsilon_r}\right)\sigma_f$$

$$= \frac{1}{2}\times30\ \mu C/m^2$$

$$= 15\ \mu C/m^2$$

Dielectric sheet ②

$$\sigma_b = \left(1-\frac{1}{3}\right)\sigma_f$$

$$= \frac{2}{3}\times30\ \mu C/m^2$$

$$= 20\ \mu C/m^2$$

Q5 See Example 4.2 in "Introduction to Electrodynamics" by David J. Griffiths.

Q6 $\oint \vec{D} \cdot d\vec{a} = Q_{fenc}$.

$$\Rightarrow \vec{D} = \frac{Q}{4\pi R^2}\hat{r}$$

$$\vec{E} = \frac{\vec{D}}{\epsilon} = \frac{Q}{4\pi\epsilon R^2}\hat{r} = \frac{Q}{4\pi\epsilon_0 K R^2}\hat{r}$$

$$\vec{P} = \epsilon_0 \chi_e \vec{E} = \frac{Q\chi_e}{4\pi K R^2}\hat{r}$$ [Linear dielectric]

Surface charge density (bound charge)

$$\sigma_b = \vec{P}\cdot\hat{r} = \frac{Q\chi_e}{4\pi K R^2} = \frac{Q(K-1)}{4\pi K R^2}$$

Q7

Q at centre

Dielectric in between R_1 & R_2

(a) Electric field

for $r < R_1$

$$\vec{E} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{r}$$

for $R_1 < r < R_2$

$$\vec{E} = \frac{Q}{4\pi\epsilon r^2}\hat{r} = \frac{Q}{4\pi\epsilon_0(1+\chi_e)r^2}\hat{r}$$

for $r > R_2$

$$\vec{E} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{r}$$

(b) Bound volume charge density $= 0$

Bound surface charge density $= (\epsilon_0\chi_e\vec{E}\cdot\hat{n})$

$$\sigma_b = \frac{Q\chi_e}{4\pi(1+\chi_e)R_2^2}$$ (outer surface)

$$= \frac{-Q\chi_e}{4\pi(1+\chi_e)R_1^2}$$ (inner surface)

(c) $\vec{\nabla}\cdot\vec{D} = \rho_f = 0$ (no free charge) for $R_1 < r_0 < R_2$