Solution to Tutorial Set-4

1. Which of the following functions cannot represent a magnetic field?
   a) $\vec{F}_1 = x^2\hat{\imath} + 3xz^2\hat{\jmath} - 2xz\hat{k}$
   b) $\vec{F}_1 = xy\hat{\imath} + yz\hat{\jmath} + 2xz\hat{k}$
   c) $\vec{F}_3 = \frac{\alpha}{(x^2+y^2)}(-y\hat{\imath} + x\hat{\jmath})$

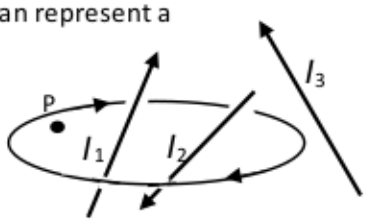
   Solution

   The necessary condition for a function to represent a magnetic field is that the divergence of the function goes to zero.

   a) $\nabla \cdot \overrightarrow{F_1} = \frac{\partial}{\partial x}x^2 + \frac{\partial}{\partial y}3xz^2 - \frac{\partial}{\partial z}2xz = 2x - 2x = 0 \Rightarrow \vec{F}_1$ can be a magnetic field.

   b) $\nabla \cdot \overrightarrow{F_2} = \frac{\partial}{\partial x}xy + \frac{\partial}{\partial y}yz + \frac{\partial}{\partial z}2xz = y + z - 2x \neq 0 \Rightarrow \overrightarrow{F_2}$ cannot represent a magnetic field.

   c) $\nabla \cdot \overrightarrow{F_3} = \frac{\partial}{\partial x}\left(\frac{-\alpha y}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{\alpha x}{x^2+y^2}\right) = \frac{\alpha y \times 2x}{(x^2+y^2)^2} - \frac{\alpha x \times 2y}{(x^2+y^2)^2} = 0 \Rightarrow \overrightarrow{F_3}$ can represent a magnetic field.



2. Three wires are carrying currents I_1, I_2 and I_3 as shown in the figure.
   a) Write down the value of $\oint \vec{B} . d\vec{l}$ over the curved path shown.
   b) Draw paths over which we will get (i) zero value and (ii) maximum positive value of $\oint \vec{B} . d\vec{l}$.
   c) What will be the values of $\nabla . \vec{B}$ and $\nabla \times \vec{B}$ at the point P?
   d) State whether the following statement is true or false: "The magnetic field along the curved path shown in the figure depends only on currents I_1 and I_2."

   Solution

   a) Applying the idea of Amperean loop $\oint \vec{B} . d\vec{l} = \mu_0(I_2 - I_1)$. *Please note the direction of integration along the loop vis a vis the directions of the currents.*
   b) The integral value will be zero if it does not enclose any of the current component and it will be maximum when it will enclose I_1 and I_3 excluding I_2 (assuming $I_1 + I_3 > I_2$) .
   c) $\nabla . \vec{B} = 0$ and $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} = 0$ since $\vec{J} = 0$ at the point P.
   d) False, the magnetic field at point depends on all the currents present.

3. A long cylindrical wire of radius *R* carries a current *I* with a volume current density of $\vec{J} = \alpha r^2 \hat{z}$ where r is the distance from the axis of the cylinder and $\hat{z}$ is the unit vector along the axis of the cylinder.
   a) Obtain the magnetic field in all regions.
   b) Obtain $\nabla . \vec{B}$ and $\nabla \times \vec{B}$ in all regions.

   Solution

   a) The magnetic field outside the wire can be evaluated by assuming an Amperean loop of radius r, where r>R, i.e $\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc} \Rightarrow B(2\pi r) = \mu_0 I \Rightarrow \vec{B} = \frac{\mu_0 I}{2\pi r}\hat{\phi}$.

Since the current density is $\vec{J} = \alpha r^2 \hat{z}$, hence inside the wire if we consider an Amperean loop of radius r then the total current enclosed by the loop is $I_{encl} = \int_0^r \vec{J} \cdot d\vec{a} = \int_0^r \alpha r^2 \times 2\pi r dr = \frac{2\pi\alpha r^4}{4} = \frac{\pi\alpha r^4}{2}$. So, the magnetic field any point inside the wire will be $B(2\pi r) = \mu_0 \frac{\pi\alpha r^4}{2} \Rightarrow \vec{B} = \frac{\mu_0 \alpha r^3}{4}\hat{\phi}$.

b) $\vec{\nabla} \cdot \vec{B} = \frac{1}{r}\frac{\partial B_\phi}{\partial \phi} = 0$ everywhere, since magnetic field does not have a ϕ dependency. This is consistent with the condition $\nabla . \vec{\boldsymbol{B}} = 0$ satisfied by the magnetic field. Now $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} = \mu_0 \alpha r^2 \hat{z}$ for r<R and zero for r>R.

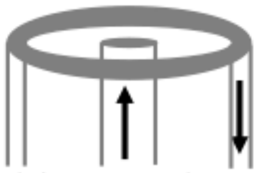
4. Consider a straight cylindrical region of thickness $(b - a)$ and having a circular cross section between inner radius a and outer radius b. A current I flows uniformly through the cross section of the cylinder.
   a) Calculate the magnetic field in all regions.
   b) Obtain $\nabla . \vec{\boldsymbol{B}}$ and $\nabla \times \vec{\boldsymbol{B}}$ in all regions.

Solution

a) Because of symmetry $\vec{B}$ will be azimuthal and depend only on r. Thus using $\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{encl}$, we find that for r<a $\vec{B} = 0$.
In the region, a<r<b, the current density is $\frac{I}{\pi(b^2 - a^2)}$. Thus, amount of current enclosed by an Amperean loop of radius r is $\frac{I\pi(r^2 - a^2)}{\pi(b^2 - a^2)}$. Hence the magnetic field $B = \frac{1}{2\pi r} \times \left(\frac{\mu_0 I\pi(r^2 - a^2)}{\pi(b^2 - a^2)}\right) = \mu_0 I \frac{r^2 - a^2}{2\pi r(b^2 - a^2)}$.
In the region r>b, the current enclosed is I hence $B = \frac{\mu_0 I}{2\pi r}$.

b) $\vec{\nabla} \cdot \vec{B}$ will be zero for all regions. In the region r<a, $\vec{\nabla} \times \vec{B} = 0$.
In the region a<r<b, $\vec{\nabla} \times \vec{B} = \frac{1}{r}\left(\frac{\partial}{\partial r}(rB)\right)\hat{z} = \frac{\mu_0 I}{\pi(b^2 - a^2)}\hat{z} = \mu_0 \vec{J}$.
In the region r>b $\vec{\nabla} \times \vec{B} = 0$.

5. Consider a coaxial configuration as shown in the figure. The inner solid cylinder carries a current in the upward direction while the outer annular cylinder (tube) carries the same current in the downward direction. Calculate the magnetic field in all regions. The radius of the inner cylinder is a and the inner and outer radii of the outer annular cylinder are b and c respectively. Calculate $\nabla . \vec{\boldsymbol{B}}$ and $\nabla \times \vec{\boldsymbol{B}}$ in all regions.



Solution

Like the previous problems, we need to consider the Amperean loop appropriately to solve the problem. One can divide the total problem in four parts,

i. r<a

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I \Rightarrow B(2\pi r) = \frac{\mu_0 I}{\pi a^2}\pi r^2 \Rightarrow B = \frac{\mu_0 I r}{2\pi a^2}$$

ii. a<r<b

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I \Rightarrow B(2\pi r) = \mu_0 I \Rightarrow B = \frac{\mu_0 I}{2\pi r}$$

iii. b<r<c

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I \Rightarrow B(2\pi r) = \mu_0 I - \frac{\mu_0 I\pi(r^2 - b^2)}{\pi(c^2 - b^2)} \Rightarrow B = \frac{\mu_0 I}{2\pi r} - \frac{\mu_0 I(r^2 - b^2)}{2\pi r(c^2 - b^2)}$$

iv. r>c

Total current in this case is zero as equal amounts of current are flowing in opposite directions. Thus B=0.

$\vec{\nabla} \cdot \vec{B} = 0$ in all regions.

However, $\vec{\nabla}\times\vec{B}$ depends on the current density at the point of evaluation.

In region (i), $\vec{\nabla}\times\vec{B} = \frac{1}{r}\left(\frac{\partial}{\partial r}(rB)\right)\hat{z} = \frac{1}{r}\left(\frac{\partial}{\partial r}\left(r\times\frac{\mu_0 I r}{2\pi a^2}\right)\right)\hat{z} = \frac{\mu_0 I}{\pi a^2}\hat{z} = \mu_0\vec{J}$.

In region (ii), $\vec{\nabla}\times\vec{B} = \frac{1}{r}\left(\frac{\partial}{\partial r}(rB)\right)\hat{z} = \frac{1}{r}\left(\frac{\partial}{\partial r}\left(r\times\frac{\mu_0 I}{2\pi r}\right)\right)\hat{z} = 0$.

In region (iii), $\vec{\nabla}\times\vec{B} = \frac{1}{r}\left(\frac{\partial}{\partial r}(rB)\right)\hat{z} = \frac{1}{r}\left(\frac{\partial}{\partial r}\left(r\times\left(\frac{\mu_0 I}{2\pi r} - \frac{\mu_0 I(r^2-b^2)}{2\pi r(c^2-b^2)}\right)\right)\right)\hat{z} = \frac{\mu_0 I}{\pi(c^2-b^2)}\hat{z} = \mu_0\vec{J}$.

In region (iv), $\vec{\nabla}\times\vec{B} = 0$ as B=0 itself.