Kepler’s law

- Law 1: Every planet moves in an elliptical orbit, with sun on one of its foci.

- Law 2: Position vector of the planet with respect to the sun, sweeps equal areas in equal times.

- Law 3: If T is the time for completing one revolution around sun, and A is the length of major axis of the ellipse, then $T^2 \propto A^3$.

- We aim to derive all these three laws based upon the mathematical theory we develop for central force motion.

Two-body problem

- Gravitational force acting on mass m_1 due to mass m_2 is $\vec{F}_{12} = -\frac{Gm_1m_2}{r_{12}^2}\hat{r}_{12}$
- It acts along the line joining two masses.
- Similarly, Coulomb force between two charges q_1 and q_2 is

$$\vec{F}_{12} = \frac{q_1q_2}{4\pi\epsilon_0 r_{12}^2}\hat{r}_{12}$$

- An idea central force can be noted as $\vec{F}(r) = f(r)\hat{r}$
- It is a one-body force depending on the coordinates of only the particle on which it acts.
- But gravity and Coulomb forces are two-body forces, of the form $\vec{F}(r_{12}) = f(r_{12})\hat{r}_{12}$.
- It is now necessary to reduce the two-body problem to a one-body problem.

We have learnt the technique of decoupling coupled equation for coupled oscillator

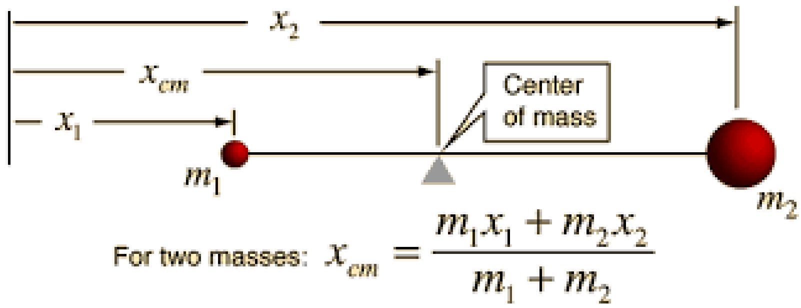
$$m\frac{d^2x_1}{dt^2} = -\frac{mg}{l}x_1 - k(x_1 - x_2)$$

$$m\frac{d^2x_2}{dt^2} = -\frac{mg}{l}x_2 + k(x_1 - x_2)$$

To solve the coupled equations, we transform var

$$X = x_1 + x_2, \qquad Y = x_1 - x_2,$$

COM & relative motion



Center of mass coordinate

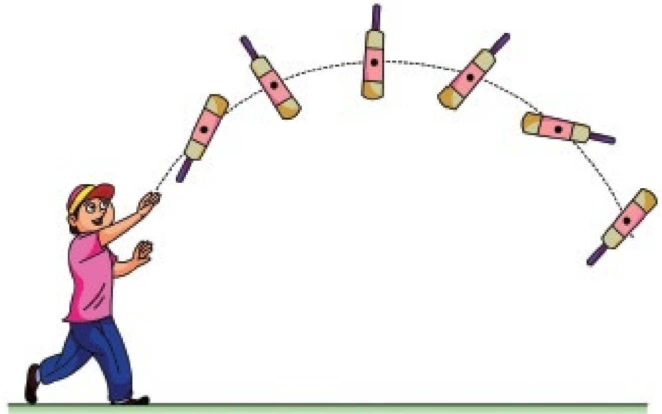
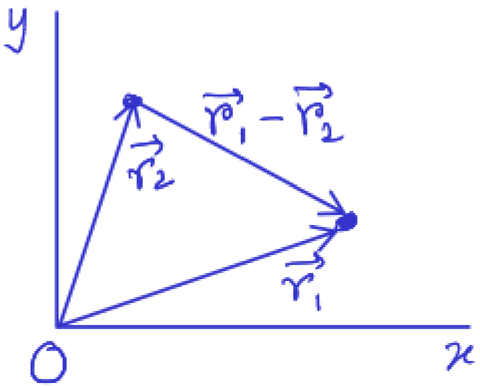
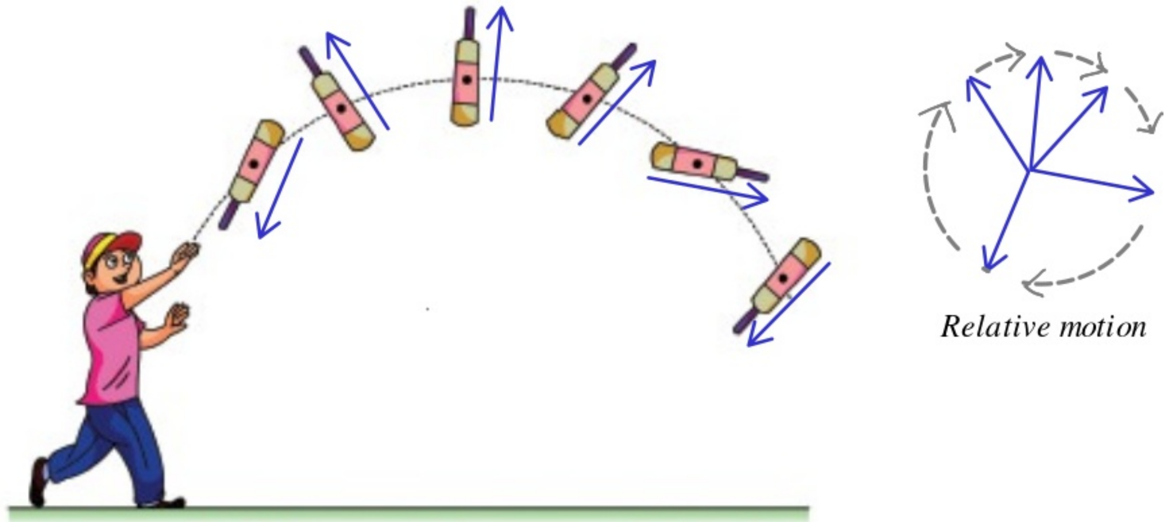


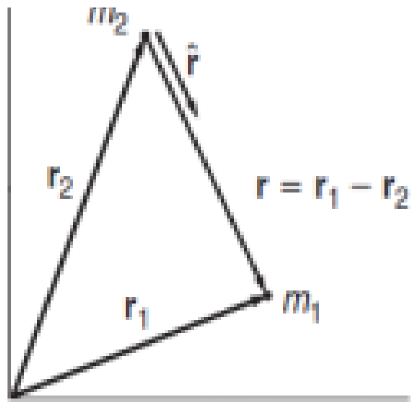
Figure 5.1 Center of mass tracing the path of a parabola



Relative coordinate



Reduction to one body problem



- Let us define $\vec{r} = \vec{r}_1 - \vec{r}_2 \Rightarrow r = |\vec{r}| = |\vec{r}_1 - \vec{r}_2|$
- Given, $\vec{F}_{12} = f(r)\hat{r}$, we have $m_1\ddot{\vec{r}}_1 = f(r)\hat{r}$ and $m_2\ddot{\vec{r}}_2 = -f(r)\hat{r}$
- Both the equations above are coupled, because both depend upon $\vec{r}_1$ and
- In order to decouple them, we replace $\vec{r}_1$ and $\vec{r}_2$ by $\vec{r} = \vec{r}_1 - \vec{r}_2$

(called relative coordinate), and center of mass coordinate $\vec{R}$.

- Since, $\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2}$, thus $\ddot{\vec{R}} = \frac{m_1\ddot{\vec{r}}_1 + m_2\ddot{\vec{r}}_2}{m_1+m_2} = \frac{f(r)\hat{r} - f(r)\hat{r}}{m_1+m_2} = 0$
- Hence, $\vec{R} = \vec{R}_0 + \vec{V}t$ where $\vec{R}_0$ is the initial location of center of mass, and $\vec{V}$ is the center of mass velocity.
- This equation physically means that the center of mass of this two-body system is moving with constant velocity, because there are no external forces on it.
- Further we can write that, $\ddot{\vec{r}}_1 - \ddot{\vec{r}}_2 = f(r)\left(\frac{1}{m_1} + \frac{1}{m_2}\right)\hat{r} \Rightarrow \ddot{\vec{r}} = \left(\frac{m_1+m_2}{m_1 m_2}\right)f(r)\hat{r} \Rightarrow \mu\ddot{\vec{r}} = f(r)\hat{r}$
- Here, $\mu = \frac{m_1 m_2}{m_1+m_2}$ is called reduced mass.

Note that final equation is in terms of only one coordinate, i.e. relative coordinate ***r.***

This equation is effectively a single particle equation of motion with mass μ moving under influence of force ***f(r).***

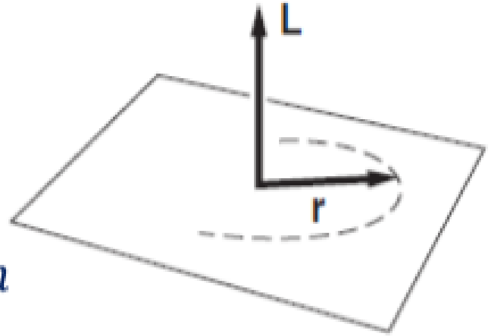
Solving relative coordinate EoM

- Thus, the two-body problem has been effectively reduced to a one-body problem
- This separation was possible only because the two-body force is central, i.e., along the line joining the two particles
- In order to solve this equation, we need to know the nature of the force, i.e., $f(r)$.
- We have already solved the equation of motion for the center-of-mass coordinate $\vec{R}$
- Therefore, once we solve the "reduced equation", we can obtain the complete solution by solving the two equations $\vec{R} = \frac{m_1\vec{r}_1+m_2\vec{r}_2}{m_1+m_2}$ and $\vec{r} = \vec{r}_1 - \vec{r}_2$.
- This leads to, $\vec{r}_1 = \vec{R} + \left(\frac{m_2}{m_1+m_2}\right)\vec{r}$ and $\vec{r}_2 = \vec{R} - \left(\frac{m_1}{m_1+m_2}\right)\vec{r}$

General properties of central force motion

- Let $\vec{L} = \vec{r} \times \vec{p}$ be angular momentum corresponding to the relative motion

  then $\frac{d\vec{L}}{dt} = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt} = \vec{v} \times \vec{p} + \vec{r} \times \vec{F}$



- However, $\vec{p} = \mu\vec{v}$, hence, $\vec{v} \times \vec{p} = 0$.
- Also, $\vec{r} \times \vec{F} = \vec{r} \times f(r)\hat{r} = 0$. This implied $\frac{d\vec{L}}{dt} = 0 \Rightarrow \vec{L} = constan$
- Thus, in case of central force motion, the angular momentum is conserved, both in direction, and magnitude.
- Conservation of angular momentum implies that the relative motion occurs in a plane.
- Direction of $\vec{L}$ is fixed, and because $\vec{r} \perp \vec{L}$, so $\vec{r}$ must be in the same plane.
- So, we can use plane polar coordinates (r, θ) to describe the motion.

Equation of motion & energy conservation - I

- We know that in plane-polar coordinate, $\vec{a} = \ddot{\vec{r}} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta}$
- Hence, $\mu\ddot{\vec{r}} = f(r)\hat{r}$ will lead to $\mu[(\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta}] = f(r)\hat{r}$.
- On comparing the coefficients of the unit vectors on both sides we obtain:

$$\mu(\ddot{r} - r\dot{\theta}^2) = f(r)$$
$$\mu(2\dot{r}\dot{\theta} + r\ddot{\theta}) = 0$$

  - By multiplying both sides of second equation by r we obtain, $\frac{d}{dt}(\mu r^2\dot{\theta}) = 0$ or $\mu r^2\dot{\theta} = L = constant$.
  - We called this constant L because it is nothing but the angular momentum of the particle about the origin.
  - Note that $L = I\omega$, with $I = \mu r^2$.
  - As the particle moves along the trajectory so that the angle θ changes by an infinitesimal amount $d\theta$, the area swept with respect to the origin is $dA = \frac{1}{2}r^2 d\theta \Rightarrow \frac{dA}{dt} = \frac{1}{2}r^2\dot{\theta} = \frac{L}{2\mu} = constant$, because L is constant.

Equation of motion & energy conservation - II

- Thus, constancy of areal velocity is a property of all central forces, not just the gravitational forces. It holds due to conservation of angular momentum.
- Kinetic energy in the plane polar coordinate can be derived as,
- $K = \frac{1}{2}\mu\vec{v}\cdot\vec{v} = \frac{1}{2}\mu\left(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta}\right)\cdot\left(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta}\right) = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\dot{\theta}^2$
- Potential energy $V(r)$ can be written as $V(r) - V(r_O) = -\int_O^r f(r)\,dr$, where r_O denotes the location of reference point.
- Total energy $E = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\dot{\theta}^2 + V(r) = constant$
- Now we have, $L = \mu r^2\dot{\theta} \Rightarrow \frac{1}{2}\mu r^2\dot{\theta}^2 = \frac{L^2}{2\mu r^2}$
- Hence, $E = \frac{1}{2}\mu\dot{r}^2 + \frac{L^2}{2\mu r^2} + V(r) = \frac{1}{2}\mu\dot{r}^2 + V_{eff}(r)$, where $V_{eff}(r) = \frac{L^2}{2\mu r^2} + V(r)$.
- In reality, $\frac{L^2}{2\mu r^2}$ is kinetic energy of the particle due to angular motion.
- But, because of its dependence on position, it can be treated as an effective potential energy.