Recap: Two-body central force motion

- For two bodies interacting via a central force of the form

  $\vec{F}_{12} = f(r)\hat{r}$, we have $m_1\ddot{\vec{r}}_1 = f(r)\hat{r}$ and $m_2\ddot{\vec{r}}_2 = -f(r)\hat{r}$

- We decouple the equations of motion in term of COM motion (in terms of ***R***) and relative motion ***r*** where $\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2}$, $r = |\vec{r}| = |\vec{r}_1 - \vec{r}_2|$

- The corresponsinf equation of motions are,

  (i) $\ddot{\vec{R}} = \frac{m_1\ddot{\vec{r}}_1 + m_2\ddot{\vec{r}}_2}{m_1+m_2} = \frac{f(r)\hat{r} - f(r)\hat{r}}{m_1+m_2} = 0$ and (ii) $\ddot{\vec{r}} = \left(\frac{m_1+m_2}{m_1 m_2}\right) f(r)\hat{r}$

- As the motion is a planar motion involving *θ*, we use plane polar coordinate system.

- We get, (i) $\mu(\ddot{r} - r\dot{\theta}^2) = f(r)$

  $\mu(2\dot{r}\dot{\theta} + r\ddot{\theta}) = 0$

  (ii) $\mu r^2\dot{\theta} = L = constant.$

  (iii) $\frac{dA}{dt} = \frac{1}{2}r^2\dot{\theta} = \frac{L}{2\mu} = constant$

  (iv) $E = \frac{1}{2}\mu\dot{r}^2 + \frac{L^2}{2\mu r^2} + V(r) = \frac{1}{2}\mu\dot{r}^2 + V_{eff}(r),$

Kepler's 1st law - I

A planet orbits the Sun in an ellipse, with the Sun at one focus of the ellipse.

- Planets are bound to sun because of gravitational force.

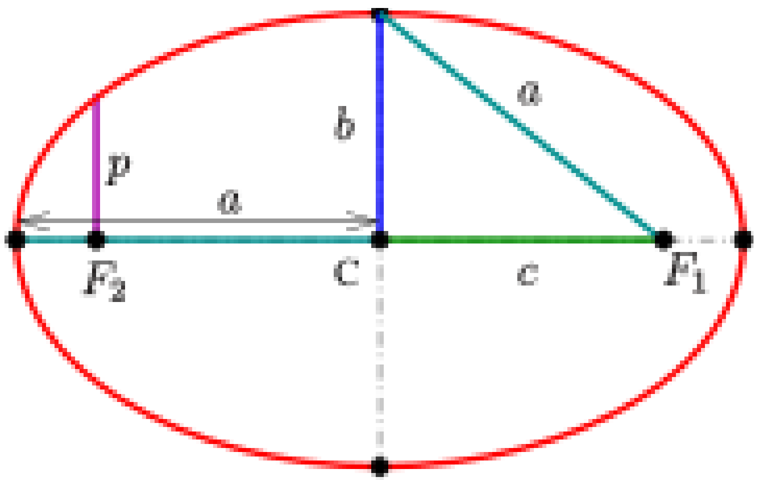
$$f(r) = -\frac{GMm}{r^2} \Rightarrow V(r) = -\frac{GMm}{r} = -\frac{C}{r}$$

- Here, $C = GMm$, where G is gravitational constant, M is mass of the Sun, and m is mass of the planet in question.

- Total energy in plane polar coordinate system, $E = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{GMm}{r}$
- *To prove 1st law, we need to show that the relation in between r and θ follows equation of ellipse.*

  Ellipse (in cartesian): $x^2/a^2 + y^2/b^2 = 1$

  Ellipse (in plain polar): $r = \frac{a(1-e^2)}{1+e\cos(\theta)}$



- a: semimajor axis, b: semi-minor axis, $b^2 + c^2 = a^2$
  c: focal length, e = c/a = eccentricity < 0 (for ellipse)
- *Where to start with? A convenient eqn involving r, θ is*

$$\mu r^2\dot{\theta} = L = constant.$$

Kepler's 1st law - II

- In $L = mr^2\dot{\theta}$ make the substitution $\rho = 1/r$, so that $\dot{\theta} = L\rho^2/m$

$$\theta = \int \frac{L}{m}\rho^2\,\mathrm{d}t = \int \frac{L}{m}\rho^2 \frac{\mathrm{d}t}{\mathrm{d}\rho}\,\mathrm{d}\rho$$

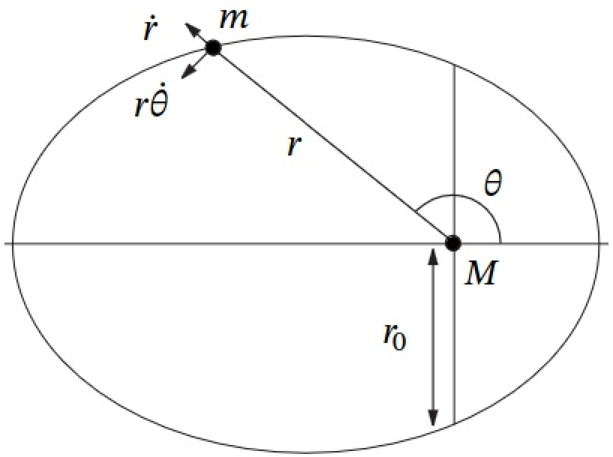
but $\dot{r} = -\frac{1}{\rho^2}\frac{\mathrm{d}\rho}{\mathrm{d}t}$ so $\theta = -\int \frac{L}{m\dot{r}}\,\mathrm{d}\rho.$

- What to substitute for *dr/dt* ? Another equation involving *dr/dt* is $E = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{GMm}{r}$

implies $\dot{r}^2 = \frac{2E}{m} + 2GM\rho - \frac{L^2}{m^2}\rho^2$

- Substituting $r_0 = \frac{L^2}{GMm^2}$ and $e^2 = 1 + \frac{2Er_0}{GMm}$ we get, $\dot{r} = \frac{L}{m}\left[\frac{e^2}{r_0^2} - \left(\rho - \frac{1}{r_0}\right)^2\right]^{1/2}$

Hence, $\theta = -\int \frac{1}{\sqrt{(e/r_0)^2 - (\rho - 1/r_0)^2}}\,\mathrm{d}\rho = \cos^{-1}\left(\frac{\rho - 1/r_0}{e/r_0}\right).$



which can be rearraged to, $r = r_0/(1 + e\cos\theta)$

Hence Kepler's 1st law is proved.

Kepler's 2nd law

<u>The line joining a planet to the Sun sweeps out equal areas in equal intervals of time.</u>

Already proved as property of motion under a central force.

- As the particle moves along the trajectory so that the angle θ changes by an infinitesimal amount $d\theta$, the area swept with respect to the origin is $dA = \frac{1}{2} r^2 d\theta \Rightarrow$ $\frac{dA}{dt} = \frac{1}{2} r^2 \dot{\theta} = \frac{L}{2\mu} = constant$, because L is constant.

Kepler's 3rd law

<u>The square of the orbital period of a planet is proportional to the cube of the semi-major axis of its orbit.</u>

- The orbital period T is simply the time taken to sweep out the total area of the ellipse

$$A_{\text{tot}} = \pi ab = \pi a^2\sqrt{1 - e^2}$$

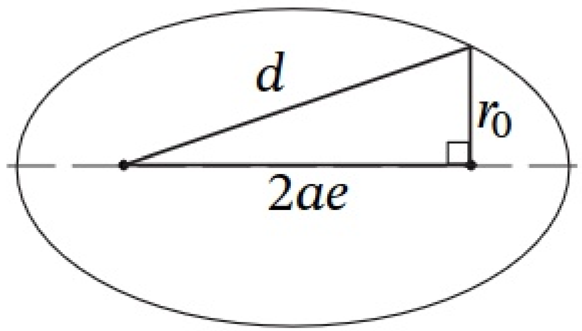
- The rate of sweeping out area is $\dot{\boldsymbol{A}} = \frac{1}{2}\boldsymbol{r} \times \dot{\boldsymbol{r}} = \frac{1}{2}\boldsymbol{r} \times \boldsymbol{v} = \frac{\boldsymbol{L}}{2m}$

Hence, $T = \dfrac{\pi ab}{L/(2m)} = \dfrac{m}{L} 2\pi a^2\sqrt{1 - e^2}$ or, $T^2 = \dfrac{m^2}{L^2} 4\pi^2 a^4 (1 - e^2).$

- Ellipse in plain polar coordinate reads: $r = \dfrac{a\left(1 - e^2\right)}{1 + e\cos(\theta)}$ and we $r = r_0/(1 + e\cos\theta)$

By comparing, we get, $r_0 = a(1 - e^2)$

- Again, we substituted $r_0 = \dfrac{L^2}{GMm^2}$ meaning, $\dfrac{m^2}{L^2} = \dfrac{1}{GMa(1 - e^2)}$



Substituting in eqn $T^2 = \dfrac{m^2}{L^2} 4\pi^2 a^4 (1 - e^2)$

we get, $T^2 = \dfrac{4\pi^2}{GM} a^3$

Hence, 3rd law is proved.