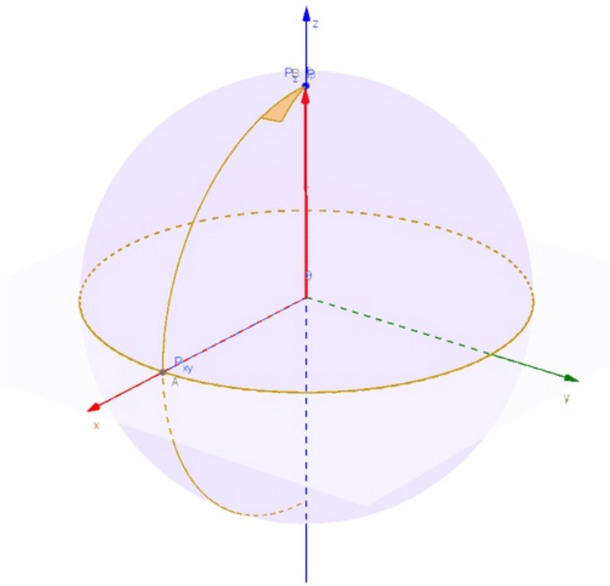
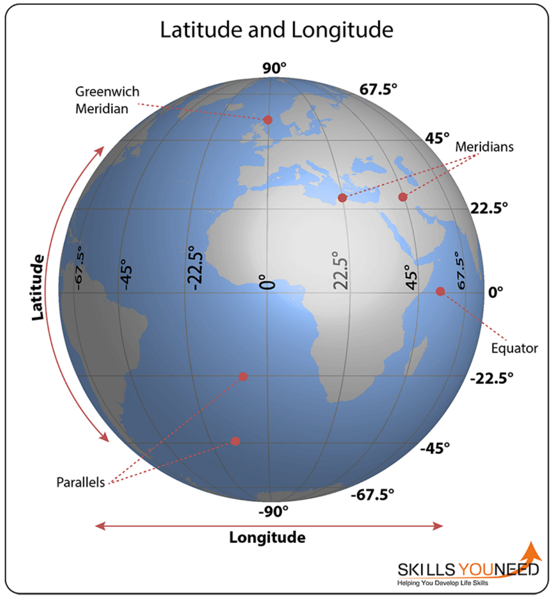
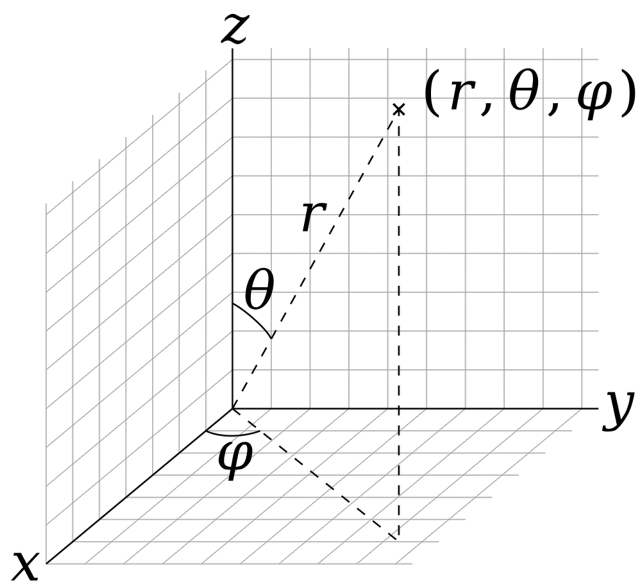
Polar coordinate system (3D)

- Starting from plain polar coordinate system which is 2D, how to cover the third dimesion; let's say, circle to sphere, or cylinder?
- **Spherical coordinate system** is used when the system or motion has spherical symmetry.
- The third dimension is described by an additional angular coordinate **φ**

$r \in [0, \infty), \theta \in [0, 2\pi)$, and $\phi \in [0, \pi]$







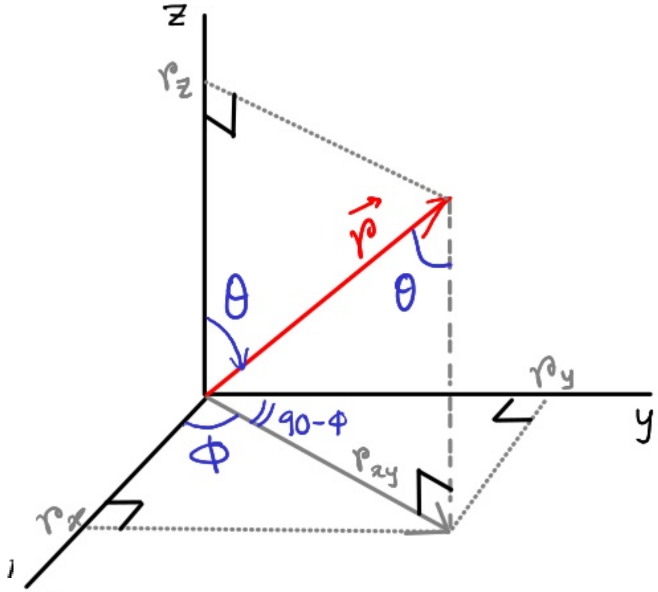
Note the axes labels and angles carefully!

Spherical coordinate system

- The relation in between cartesian and spherical coordinate system:

1. $|\vec{r}_{xy}| = |\vec{r}| \sin\theta$
2. $r_x = r_{xy} \cos\phi$
   $= |\vec{r}| \sin\theta \cos\phi$
3. $r_y = r_{xy} \sin\phi$
   $= |\vec{r}| \sin\theta \sin\phi$
4. $r_z = |\vec{r}| \cos\theta$

$$r^2 = x^2 + y^2 + z^2$$
$$\phi = \tan^{-1}(y/x)$$
$$\theta = \cos^{-1}(z/r)$$

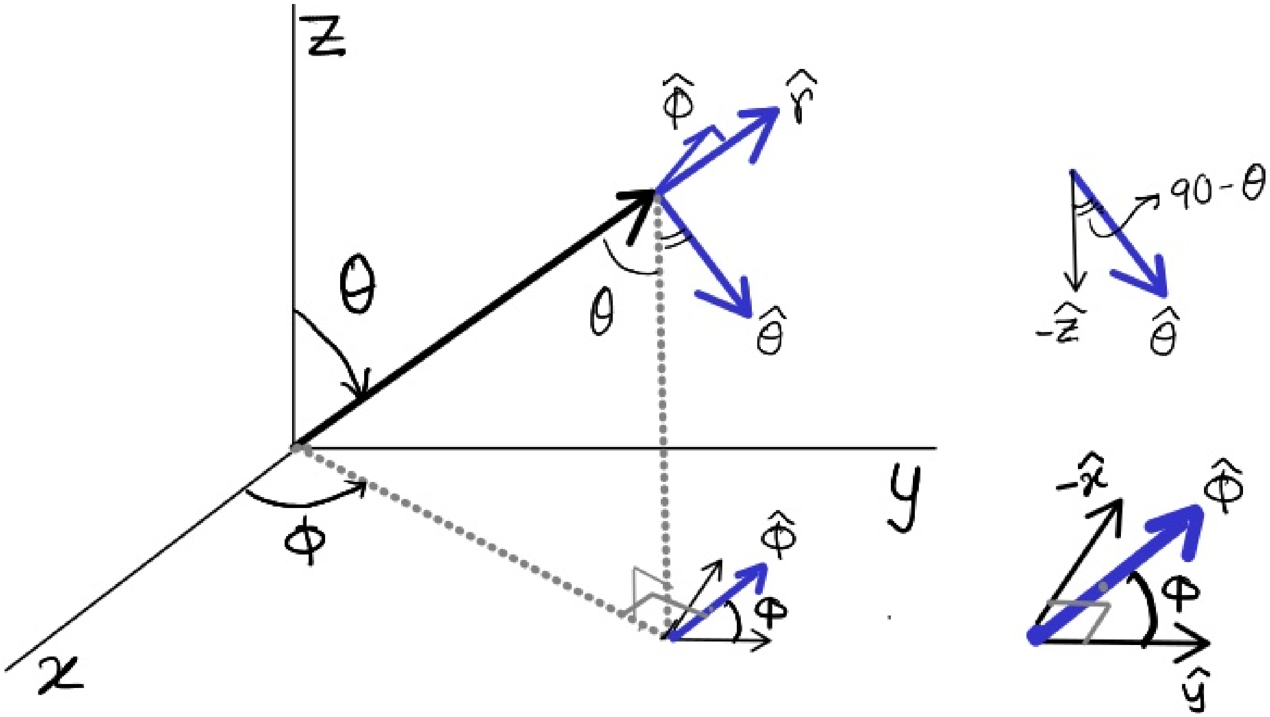


- Radius vector: $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$
  $$= r\sin\theta\cos\phi\,\hat{i} + r\sin\theta\sin\phi\,\hat{j} + r\cos\theta\,\hat{k}$$

- Unit vector: $\hat{r} = \dfrac{\vec{r}}{|\vec{r}|} = \dfrac{r_x}{|\vec{r}|}\hat{i} + \dfrac{r_y}{|\vec{r}|}\hat{j} + \dfrac{r_z}{|\vec{r}|}\hat{k}$
  $$= \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}$$

Spherical coordinate system

- Conversion from spherical to Cartesian coordinate system:



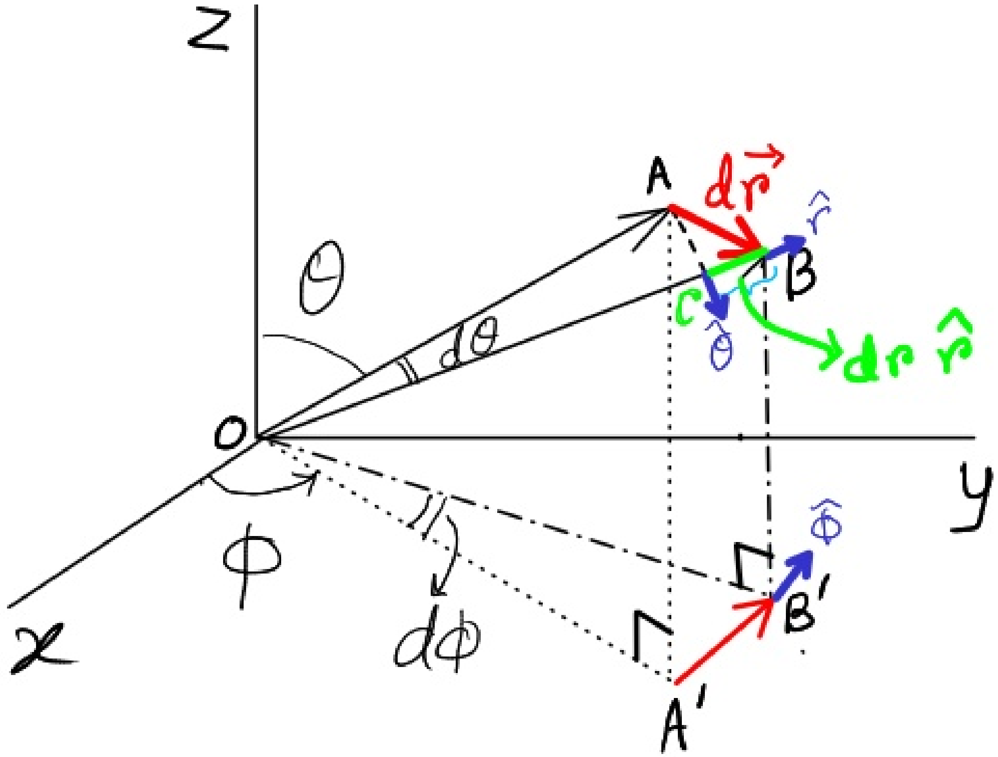
$$\hat{r} = \frac{\vec{r}}{|\vec{r}|} = \sin\theta\cos\varphi\,\hat{\mathbf{x}} + \sin\theta\sin\varphi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}}$$

$$\hat{\boldsymbol{\varphi}} = -\sin\varphi\,\hat{\mathbf{x}} + \cos\varphi\,\hat{\mathbf{y}}$$

$$\hat{\boldsymbol{\theta}} = \cos\theta\cos\varphi\,\hat{\mathbf{x}} + \cos\theta\sin\varphi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}}$$

Spherical coordinate system

- Line element for infinitesimal displacement from *(r, θ, φ)* to *(r+dr, θ+dθ, φ+dφ)*



$$\overrightarrow{CB} = dr\,\hat{r}$$
$$\overrightarrow{AC} = r\,d\theta\,\hat{\theta}$$

$$\angle OAA' = \theta$$
$$OA' = r\sin\theta$$
$$\overrightarrow{A'B'} = r\sin\theta\,d\phi\,\hat{\phi}$$

$$\mathbf{dr} = \mathrm{d}r\,\hat{\mathbf{r}} + r\,\mathrm{d}\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,\mathrm{d}\varphi\,\hat{\boldsymbol{\varphi}}$$

Spherical coordinate system

- In general, line element can be written as,

$$\mathrm{d}\mathbf{r} = \sum_i \frac{\partial \mathbf{r}}{\partial x_i}\,\mathrm{d}x_i = \sum_i \left|\frac{\partial \mathbf{r}}{\partial x_i}\right| \frac{\frac{\partial \mathbf{r}}{\partial x_i}}{\left|\frac{\partial \mathbf{r}}{\partial x_i}\right|}\,\mathrm{d}x_i = \sum_i \left|\frac{\partial \mathbf{r}}{\partial x_i}\right| \mathrm{d}x_i\,\hat{\boldsymbol{x}}_i$$

$$\mathbf{r} = \begin{bmatrix} r\sin\theta\,\cos\varphi \\ r\sin\theta\,\sin\varphi \\ r\cos\theta \end{bmatrix}.$$

Thus,

$$\frac{\partial \mathbf{r}}{\partial r} = \begin{bmatrix} \sin\theta\,\cos\varphi \\ \sin\theta\,\sin\varphi \\ \cos\theta \end{bmatrix} = \hat{\mathbf{r}}, \quad \frac{\partial \mathbf{r}}{\partial \theta} = \begin{bmatrix} r\cos\theta\,\cos\varphi \\ r\cos\theta\,\sin\varphi \\ -r\sin\theta \end{bmatrix} = r\,\hat{\boldsymbol{\theta}}, \quad \frac{\partial \mathbf{r}}{\partial \varphi} = \begin{bmatrix} -r\sin\theta\,\sin\varphi \\ r\sin\theta\,\cos\varphi \\ 0 \end{bmatrix} = r\sin\theta\,\hat{\boldsymbol{\varphi}}$$

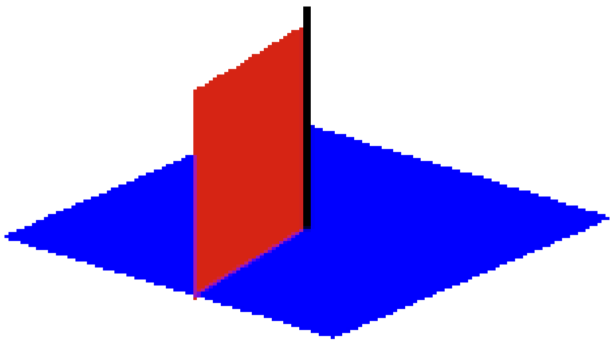
The desired coefficients are the magnitudes of these vectors:[5]

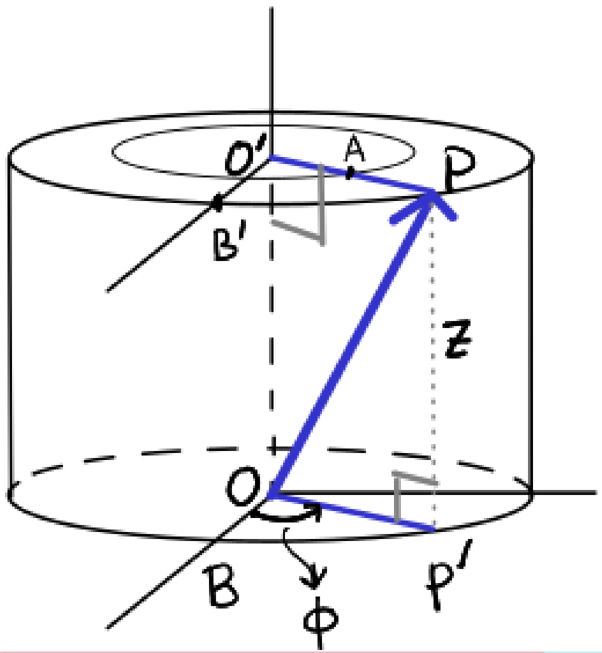
$$\left|\frac{\partial \mathbf{r}}{\partial r}\right| = 1, \quad \left|\frac{\partial \mathbf{r}}{\partial \theta}\right| = r, \quad \left|\frac{\partial \mathbf{r}}{\partial \varphi}\right| = r\sin\theta.$$

Cylindrical coordinate system

- Cylindrical coordinate system: Convenient to describe a point which moves on a surface of a cylinder.
- A point (in 3D) is described by 3 coordinates:

a) radius of the cylinder (r or ρ)
b) azimuthal angle from x-axis (θ or φ)
c) height of the point from xy plane (z)





$$P(r,\phi,z)\ ,\ A(r',\phi,z)$$

$$OP' = O'P = r$$

$$\angle BOP' = B'O'P = B'O'A = \phi$$

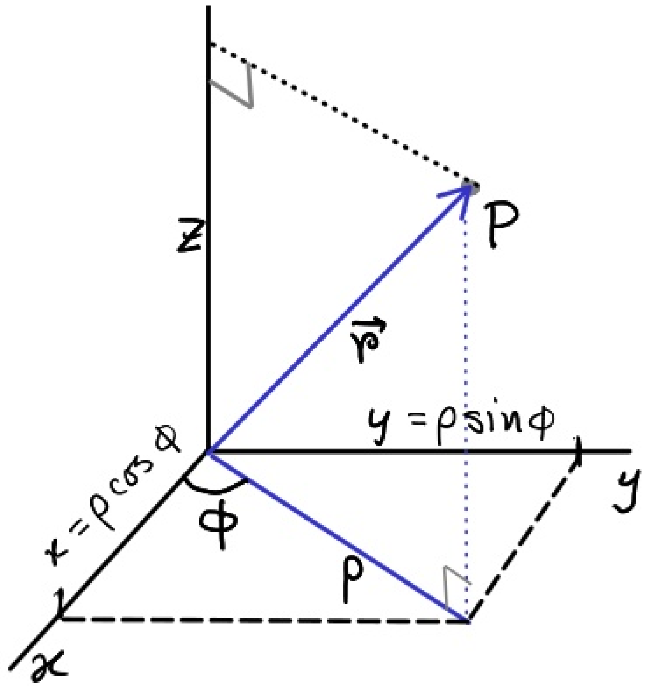
$$OO' = P'P = z$$

Conversion: Cylindrical to Cartesian

- The vector **r** can be expressed as

$$\vec{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$$
$$= \rho cos\phi\,\hat{i} + \rho\, sin\phi\,\hat{j} + z\,\hat{k}$$

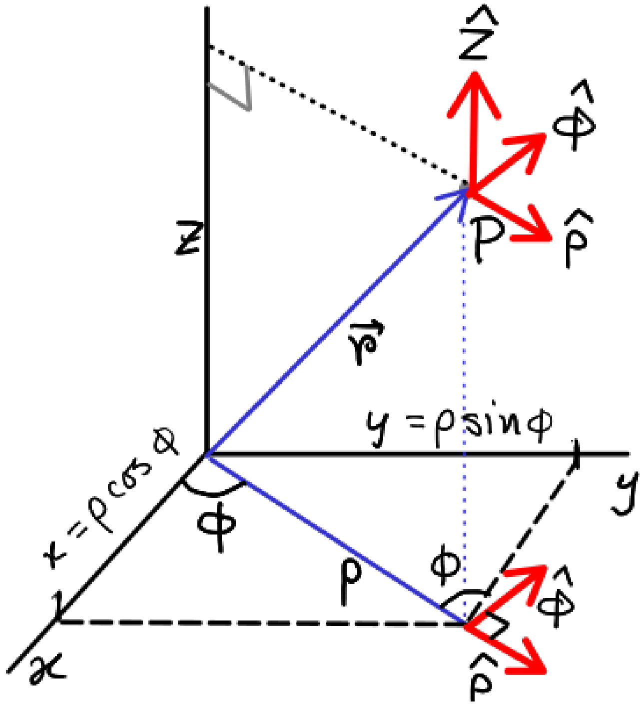
From Cartesian to Cylindrical	From Cylindrical to Cartesian
$\rho = \sqrt{x^2 + y^2}$	$x = \rho cos\phi$
$\phi = tan^{-1}\frac{y}{x}$	$y = \rho sin\phi$
$z = z$	$z = z$



Conversion: Cylindrical to Cartesian

- The vector **r** can be expressed as

$$\vec{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$$
$$= \rho\cos\phi\,\hat{i} + \rho\sin\phi\,\hat{j} + z\,\hat{k}$$

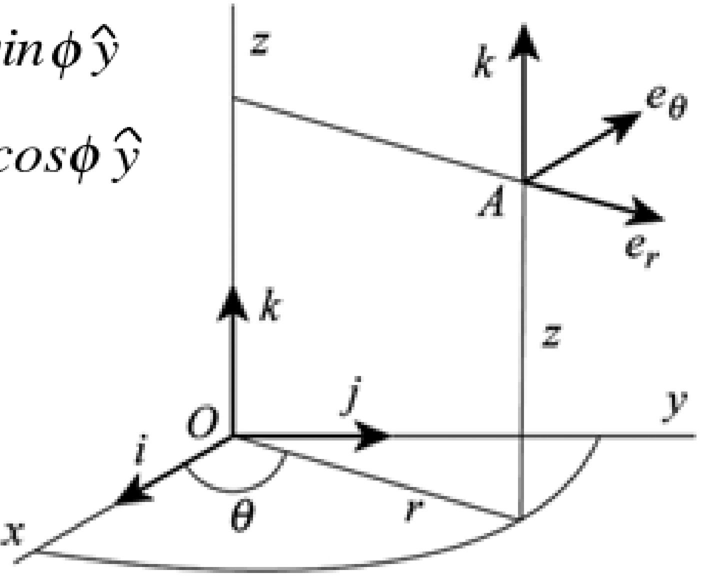


From Cartesian to Cylindrical	From Cylindrical to Cartesian
$\rho = \sqrt{x^2 + y^2}$	$x = \rho\cos\phi$
$\phi = \tan^{-1}\frac{y}{x}$	$y = \rho\sin\phi$
$z = z$	$z = z$

$$\hat{\rho} = \frac{x}{\rho}\hat{x} + \frac{y}{\rho}\hat{y} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y}$$

$$\phi = -\frac{y}{\rho}\hat{x} + \frac{x}{\rho}\hat{y} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$$

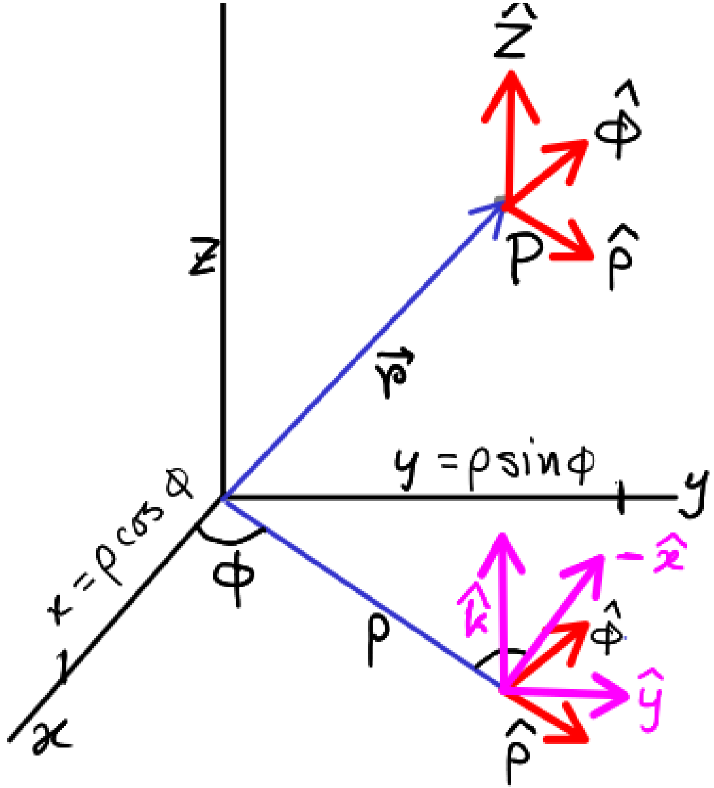
$$\hat{z} = \hat{z}$$



Conversion: Cylindrical to Cartesian

- The vector **r** can be expressed as

$$\vec{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$$
$$= \rho\cos\phi\,\hat{i} + \rho\sin\phi\,\hat{j} + z\,\hat{k}$$



From Cartesian to Cylindrical	From Cylindrical to Cartesian
$\rho = \sqrt{x^2 + y^2}$	$x = \rho\cos\phi$
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$z = z$	$z = z$

$$\hat{\rho} = \frac{x}{\rho}\hat{x} + \frac{y}{\rho}\hat{y} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y}$$

$$\phi = -\frac{y}{\rho}\hat{x} + \frac{x}{\rho}\hat{y} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$$

$$\hat{z} = \hat{z}$$

$$\hat{x} = \cos\phi\,\hat{\rho} - \sin\phi\,\hat{\phi}$$
$$\hat{y} = \sin\phi\,\hat{\rho} + \cos\phi\,\hat{\phi}$$
$$\hat{z} = \hat{z}$$

	$\hat{\rho}$	$\hat{\phi}$	$\hat{z}$
$\hat{x}$	$\cos\phi$	$-\sin\phi$	0
$\hat{y}$	$\sin\phi$	$\cos\phi$	0
$\hat{z}$	0	0	1

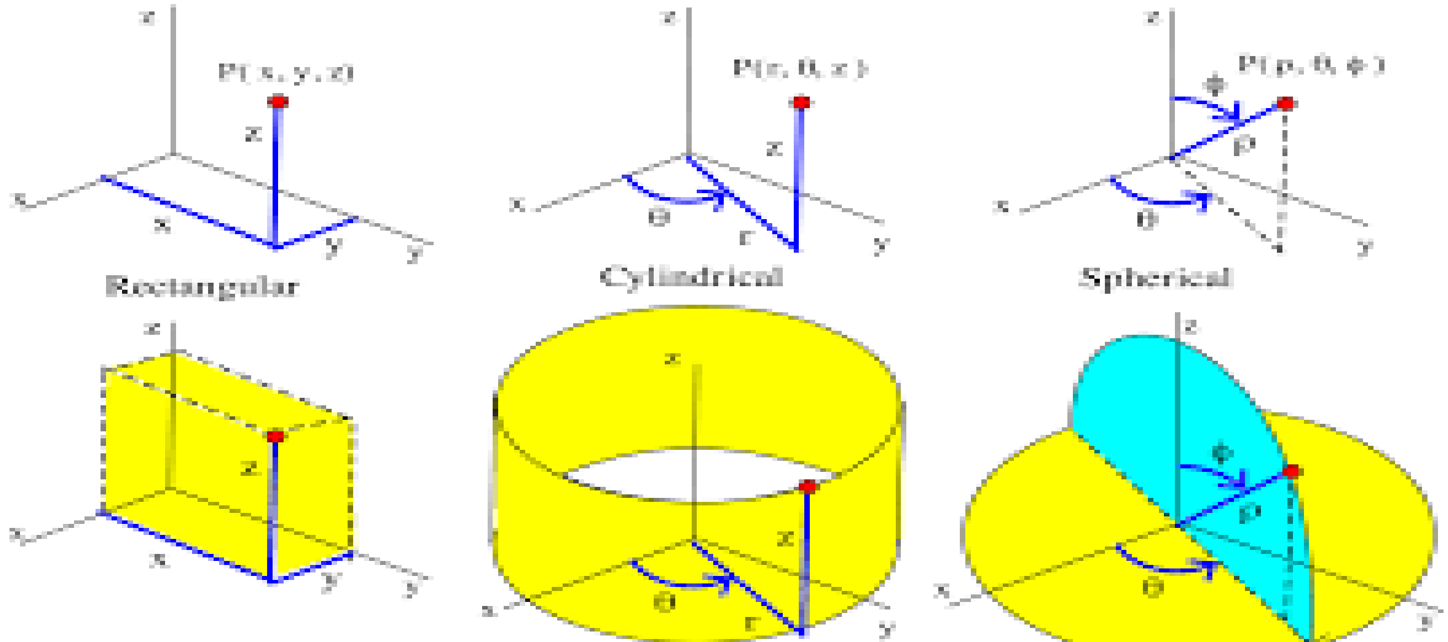

Fig. 2

	$\hat{\rho}$	$\hat{\phi}$	$\hat{z}$
$\hat{x}$	$\cos\phi$	$-\sin\phi$	0
$\hat{y}$	$\sin\phi$	$\cos\phi$	0
$\hat{z}$	0	0	1

	$\hat{r}$	$\hat{\theta}$	$\hat{\phi}$
$\hat{x}$	$\sin\theta\cos\phi$	$\cos\theta\cos\phi$	$-\sin\phi$
$\hat{y}$	$\sin\theta\sin\phi$	$\cos\theta\sin\phi$	$\cos\phi$
$\hat{z}$	$\cos\theta$	$-\sin\theta$	0