Product of vectors

- **Dot product in cartesian oordinate system:**
- Because these basis vectors are perpendicular to each other, they satisfy

$$\hat{\imath}\cdot\hat{\imath} = \hat{\jmath}\cdot\hat{\jmath} = \hat{k}\cdot\hat{k} = 1$$

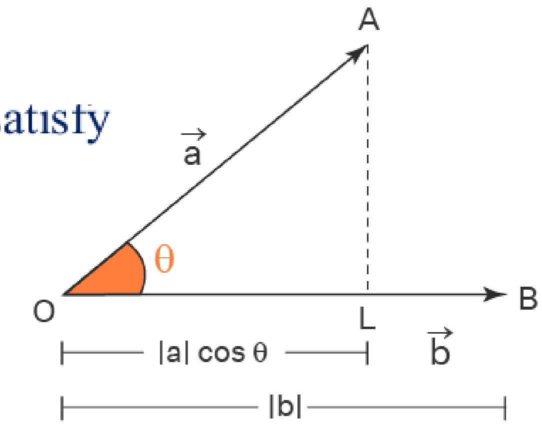
$$\hat{\imath}\cdot\hat{\jmath} = \hat{\jmath}\cdot\hat{k} = \hat{k}\cdot\hat{\imath} = 0$$

$$\vec{a} = (a_1\vec{i} + a_2\vec{j} + a_3\vec{k})$$

$$\vec{b} = (b_1\vec{i} + b_2\vec{j} + b_3\vec{k})$$

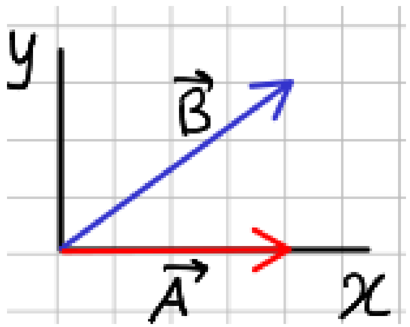
$$\vec{a}\cdot\vec{b} = (a_1\vec{i} + a_2\vec{j} + a_3\vec{k}).(b_1\vec{i} + b_2\vec{j} + b_3\vec{k})$$

$$= a_1b_1 + a_2b_2 + a_3b_3$$



a . b = |a|.|b| cos θ

- Example:



$$\vec{A} = 4\hat{i}\ ,\ \vec{B} = 4\hat{i} + 3\hat{j}$$
$$|\vec{A}| = \sqrt{4^2} = 4$$
$$|\vec{B}| = \sqrt{4^2 + 3^2} = 5$$

$$\tan\theta = \frac{\Delta y}{\Delta x} = \frac{3}{4}$$
$$\Rightarrow \theta = \tan^{-1}\frac{3}{4} = 36.87^o$$

$$\vec{A}.\vec{B} = (4\hat{i}).(4\hat{i} + 3\hat{j})$$
$$= (4\hat{i}).(4\hat{i}) = 16$$

$$\cos\theta = \frac{\vec{A}.\vec{B}}{|\vec{A}||\vec{B}|}$$

$$\theta = cos^{-1}\frac{16}{5.4} = \cos^{-1}\frac{4}{5} = 36.87^o$$

Product of vectors

- Cross product in cartesian oordinate system:

Vector Cross Product Formula

$$\vec{a} \times \vec{b} = |\vec{a}|\,|\vec{b}| \sin\theta \,\hat{n}$$

$$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k} \;, \quad \vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$$

$$\vec{a}\times\vec{b} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \times (b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= i\,(a_2b_3 - a_3b_2) - j\,(a_1b_3 - a_3b_1) + k\,(a_1b_2 - a_2b_1)$$

$$\hat{\imath} \times \hat{\imath} = \hat{\jmath} \times \hat{\jmath} = \hat{k} \times \hat{k} = 0$$

$$\hat{\imath} \times \hat{\jmath} = \hat{k};\; \hat{\jmath} \times \hat{k} = \hat{\imath};\; \hat{k} \times \hat{\imath} = \hat{\jmath}$$

Cross product

- $$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

$$= \hat{i}(a_y b_z - a_z b_y) + \hat{j}(a_z b_x - a_x b_z) + \hat{k}(a_x b_y - a_y b_x)$$

- Example:

$$\vec{A} \times \vec{B} = \begin{vmatrix} i & j & k \\ 4 & 0 & 0 \\ 4 & 3 & 0 \end{vmatrix} = \hat{i}(0\times0-0\times3)+\hat{j}(0\times4-4\times0)+\hat{k}(4\times3-0\times4) = 12\hat{k}$$

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Gradient, Divergence, Curl in Cartesian coordinate

- $\vec{\nabla}$ is a vector differential operator that operates on scalar or vector
- Del operator in Cartesian coordinate system:

$$\vec{\nabla} \equiv \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

- **Gradient**: Del operating on a scalar *f* gives a vector.

$$\vec{\nabla} f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$$

- **Divergence**: Del operating on a vector through dot product gives a scalar.

$$\vec{\nabla}.\vec{A} = (\hat{i}\frac{\partial}{\partial x}).(\hat{i}A_x) + (\hat{j}\frac{\partial}{\partial y}).(\hat{j}A_y) + (\hat{k}\frac{\partial}{\partial z}).(\hat{k}A_z)$$
$$= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

- **Curl**: Del operating on a vector through cross product gives a vector.

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \hat{i}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{j}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{k}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

Gradient: Physical meaning

- The gradient is always pointing in the direction of the function's steepest increase.

Proof: If I take a unit vector ***w*** and dot product with gradient: $\vec{\nabla} f \,.\, \hat{w} = |\vec{\nabla} f| \cos\theta$

will be maximum when θ will be 0. So gradient gives the direction of maximum change.

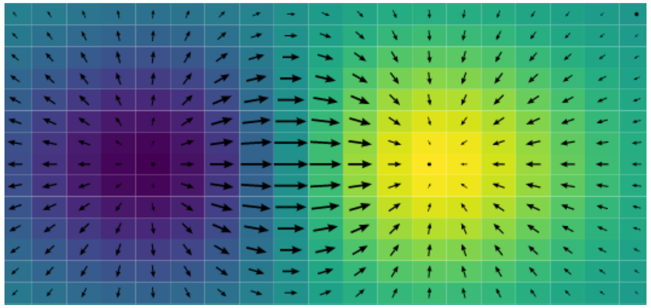
- The gradient is perpendicular to the surface with constant *f(x,y,z)* (called "level surface")

Proof: You can also visualize the gradient using the *level surfaces* on which $f(x, y, z) = \text{const.}$ (In two dimensions there is the analogous concept of *level curves*, on which $f(x, y) = \text{const.}$) Consider a small displacement $d\vec{\mathbf{r}}$ that lies on the level surface, that is, start at a point on the level surface, and move along the surface. Then f doesn't change in that direction, so $df = 0$. But then

$$0 = df = \vec{\boldsymbol{\nabla}} f \cdot d\vec{\mathbf{r}} = 0 \qquad \vec{\boldsymbol{\nabla}} f \perp \{f(x, y, z) = \text{const}\}$$

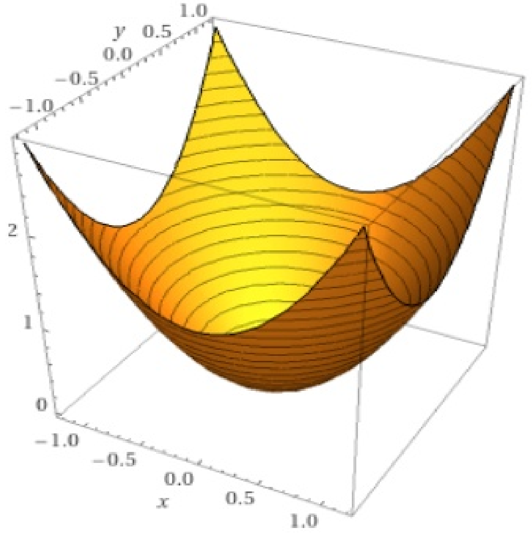
so that $\vec{\boldsymbol{\nabla}} f$ is perpendicular to $d\vec{\mathbf{r}}$. Since this argument works for *any* vector displacement $d\vec{\mathbf{r}}$ in the surface, $\vec{\boldsymbol{\nabla}} f$ must be perpendicular to the level surface.

- The gradient's strength indicates how quickly the function is changing in that direction.
- When the gradient is zero, a critical point (the maximum, minimum, or saddle point) is present.

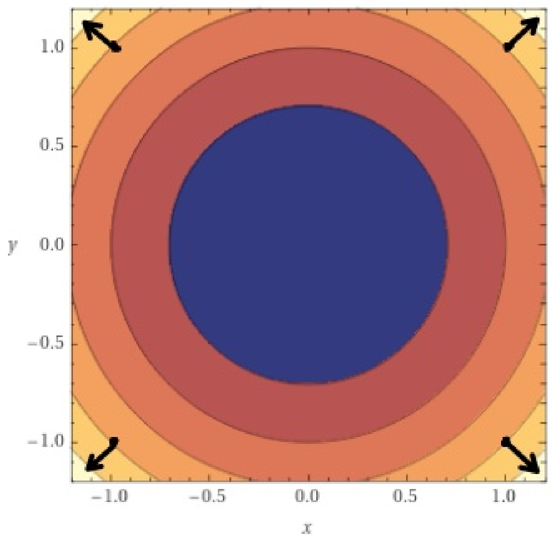


Gradient: Example I

$$f(x,y) = x^2 + y^2$$
$$\vec{\nabla} f(x,y) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y}\right) f(x,y)$$
$$= \frac{\partial f(x,y)}{\partial x}\,\hat{i} + \frac{\partial f(x,y)}{\partial y}\,\hat{j}$$
$$= 2x\,\hat{i} + 2y\,\hat{j}$$

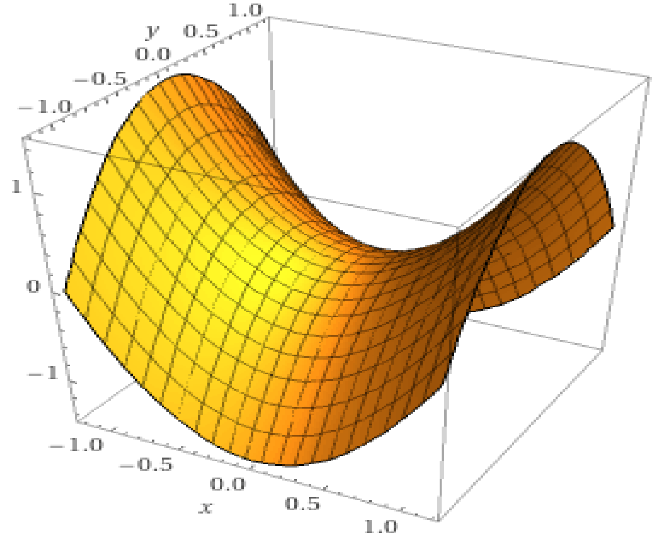


$$\vec{\nabla} f(x,y)|_{(x=0,y=0)} = 0$$
$$\vec{\nabla} f(x,y)|_{(x=1,y=1)} = 2\hat{i} + 2\hat{j}$$
$$\vec{\nabla} f(x,y)|_{(x=1,y=-1)} = 2\hat{i} - 2\hat{j}$$
$$\vec{\nabla} f(x,y)|_{(x=-1,y=1)} = -2\hat{i} + 2\hat{j}$$
$$\vec{\nabla} f(x,y)|_{(x=-1,y=-1)} = -2\hat{i} - 2\hat{j}$$

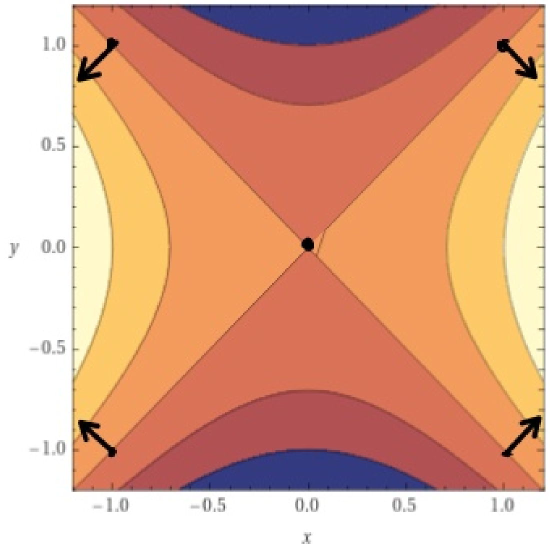


Gradient: Example II

$$f(x,y) = x^2 - y^2$$
$$\vec{\nabla} f(x,y) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y}\right) f(x,y)$$
$$= \frac{\partial f(x,y)}{\partial x}\,\hat{i} - \frac{\partial f(x,y)}{\partial y}\,\hat{j}$$
$$= 2x\,\hat{i} - 2y\,\hat{j}$$



$$\vec{\nabla} f(x,y)|_{(x=0,y=0)} = 0$$
$$\vec{\nabla} f(x,y)|_{(x=1,y=1)} = 2\hat{i} - 2\hat{j}$$
$$\vec{\nabla} f(x,y)|_{(x=1,y=-1)} = 2\hat{i} + 2\hat{j}$$
$$\vec{\nabla} f(x,y)|_{(x=-1,y=1)} = -2\hat{i} - 2\hat{j}$$
$$\vec{\nabla} f(x,y)|_{(x=-1,y=-1)} = -2\hat{i} + 2\hat{j}$$



Formulae in different coordinate system

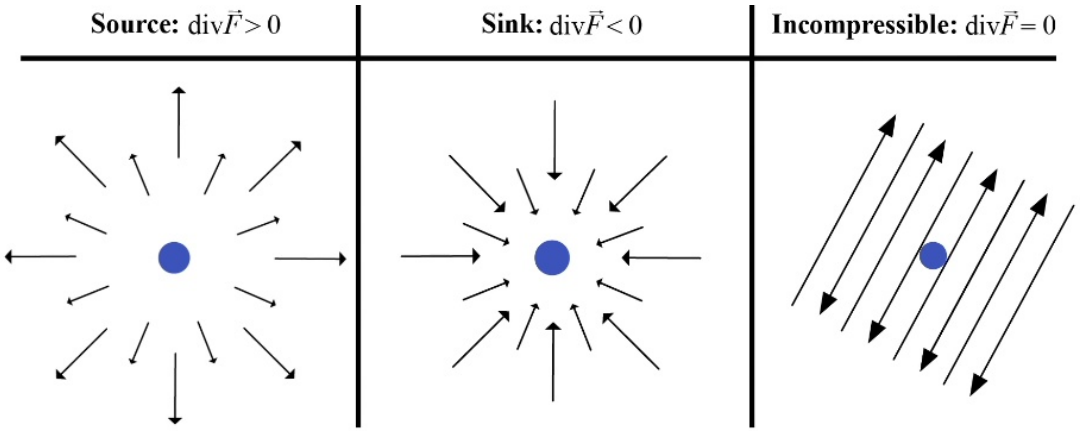
Table with the del operator in cartesian, cylindrical and spherical coordinates

Operation	Cartesian coordinates (x, y, z)	Cylindrical coordinates (ρ, φ, z)	Spherical coordinates (r, θ, φ), where θ is the polar angle and φ is the azimuthal angle
Vector field $\mathbf{A}$	$A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}$	$A_\rho\hat{\boldsymbol{\rho}} + A_\varphi\hat{\boldsymbol{\varphi}} + A_z\hat{\mathbf{z}}$	$A_r\hat{\mathbf{r}} + A_\theta\hat{\boldsymbol{\theta}} + A_\varphi\hat{\boldsymbol{\varphi}}$
Gradient ∇f[1]	$\frac{\partial f}{\partial x}\hat{\mathbf{x}} + \frac{\partial f}{\partial y}\hat{\mathbf{y}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}}$	$\frac{\partial f}{\partial \rho}\hat{\boldsymbol{\rho}} + \frac{1}{\rho}\frac{\partial f}{\partial \varphi}\hat{\boldsymbol{\varphi}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}}$	$\frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \varphi}\hat{\boldsymbol{\varphi}}$
Divergence $\nabla \cdot \mathbf{A}$[1]	$\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$	$\frac{1}{\rho}\frac{\partial(\rho A_\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial A_\varphi}{\partial \varphi} + \frac{\partial A_z}{\partial z}$	$\frac{1}{r^2}\frac{\partial(r^2 A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(A_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial \varphi}$

Divergence: Physical meaning

- Divergence measures the tendency of a vector field to disperse or collect at a point. It is a local measure of its "outgoingness".

$$
\begin{aligned}
\vec{\nabla}.\vec{A} &= (\hat{i}\frac{\partial}{\partial x}).(\hat{i}A_x) + (\hat{j}\frac{\partial}{\partial y}).(\hat{j}A_y) + (\hat{k}\frac{\partial}{\partial z}).(\hat{k}A_z) \\
&= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}
\end{aligned}
$$

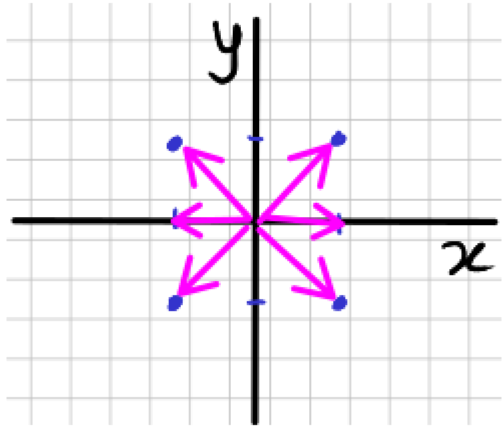


Divergence: Example

- Outgoing: div A>0

$$\vec{A}(x,y) = x\hat{i} + y\hat{j}$$

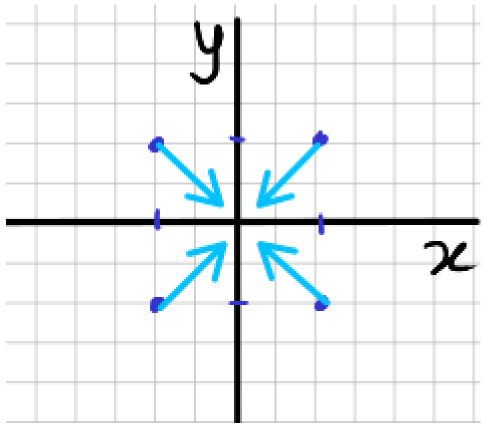
$$\vec{\nabla}.\vec{A} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j}\right).(x\hat{i} + y\hat{j}) = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} = 1 + 1 = 2$$



- Incoming: div A <0

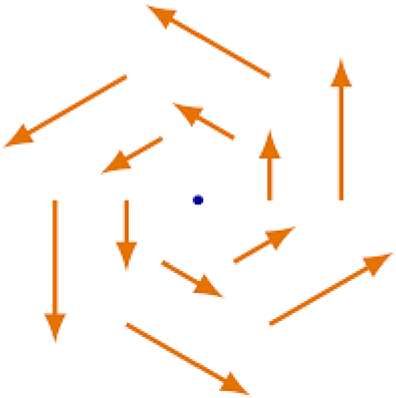
$$\vec{A}(x,y) = -x\hat{i} - y\hat{j}$$

$$\vec{\nabla}.\vec{A} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j}\right).(-x\hat{i} - y\hat{j}) = -\frac{\partial x}{\partial x} - \frac{\partial y}{\partial y} = -2$$

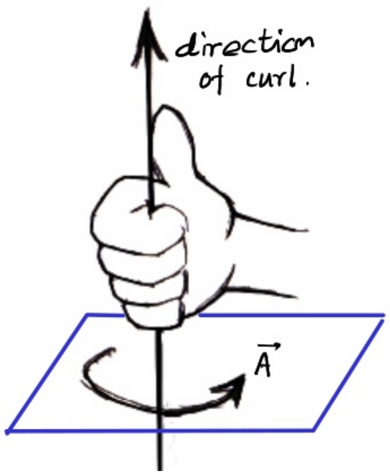


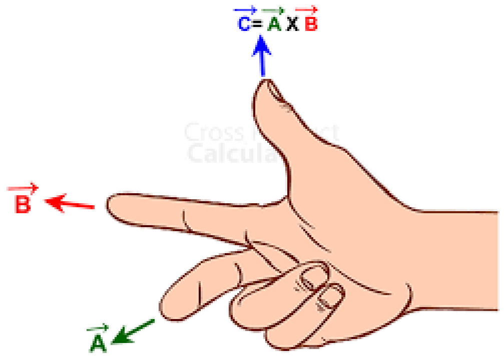
Curl: Physical meaning

- Curl is an operation, which when applied to a vector field, quantifies the circulation.
  The magnitude of the curl measures how much the vector field is swirling, the direction indicates the axis around which it tends to swirl.



$$\nabla \times \mathbf{v} = \odot$$



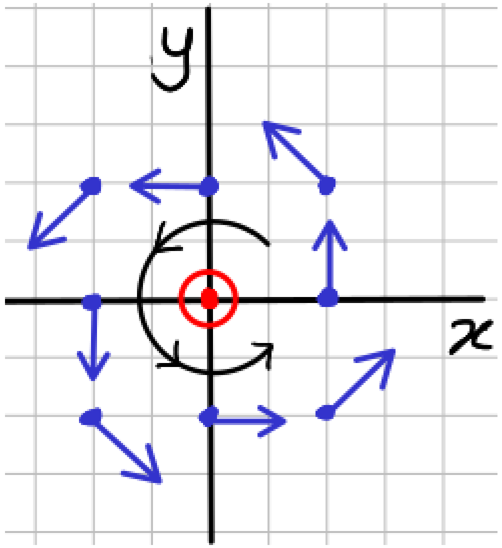


Curl: Example

- Let's take a vector ***A***:

$$\vec{A}(x, y) = -y\,\hat{i} + x\,\hat{j}$$

$$\vec{A}(1,0) = \hat{j}$$
$$\vec{A}(1,1) = -\hat{i} + \hat{j}$$
$$\vec{A}(0,1) = -\hat{i}$$
$$\vec{A}(-1,1) = -\hat{i} - \hat{j}$$
$$\ldots$$



Curl: Example

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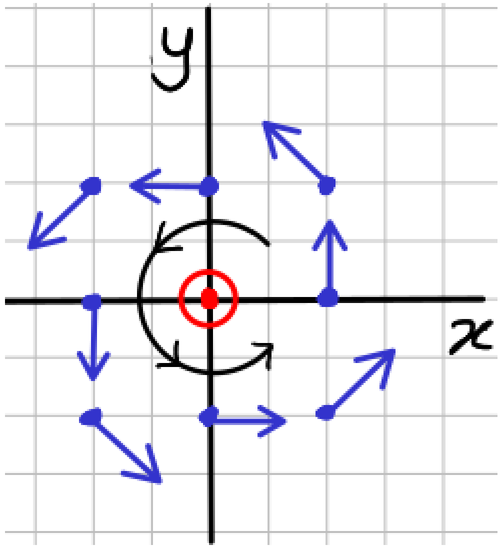
$$\vec{A}(-1,1) = -\hat{i} - \hat{j}$$
$$\ldots$$



$$curl\ \vec{A} = \vec{\nabla} \times \vec{A} = \hat{i}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{j} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{k}$$
$$= \hat{i}\left(\frac{\partial (0)}{\partial y} - \frac{\partial x}{\partial z}\right) + \left(\frac{\partial (-y)}{\partial z} - \frac{\partial (0)}{\partial x}\right)\hat{j} + \left(\frac{\partial x}{\partial x} - \frac{\partial (-y)}{\partial y}\right)\hat{k}$$
$$= 2\ \hat{k}$$