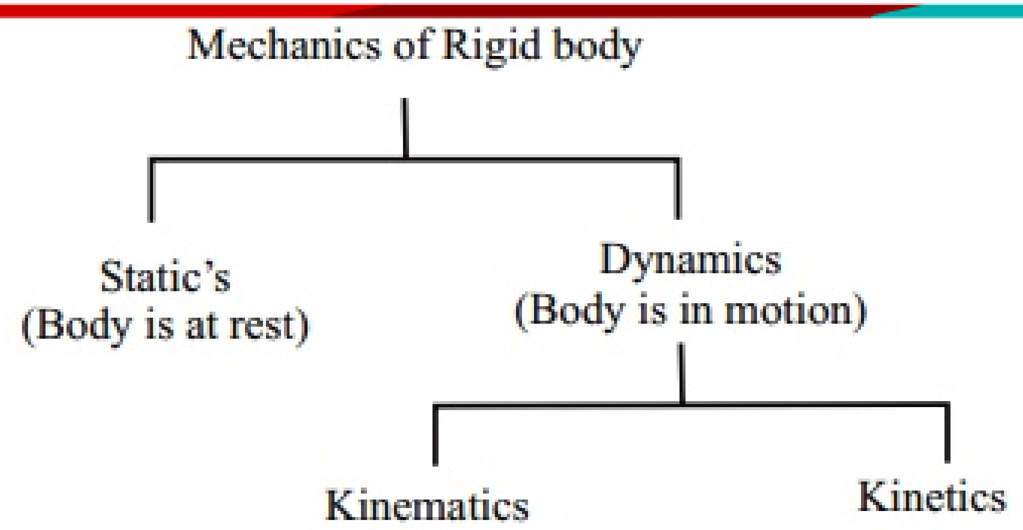
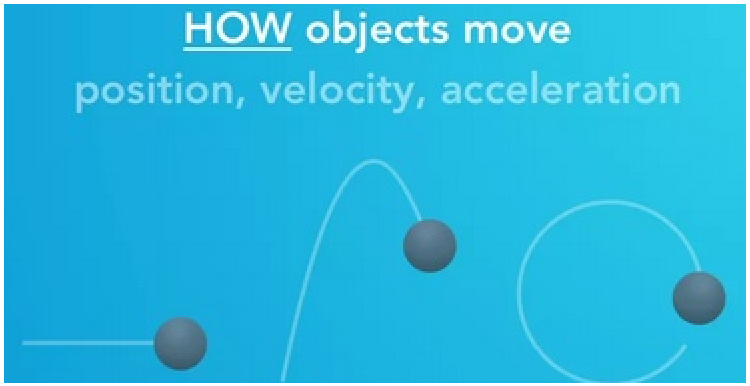
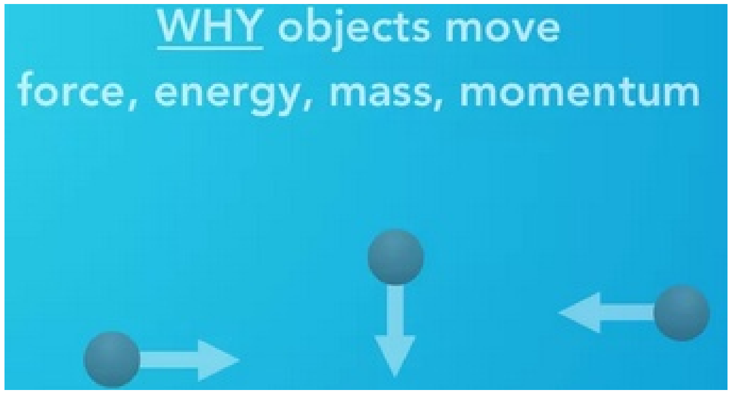
# Mechanics




*1- __Kinematics__:* **study of the motion of particles/rigid bodies (relate displacement, velocity, acceleration, and time, without reference to the cause of the motion).**

- *2- __Kinetics__:* **study of the forces acting on the particles/rigid bodies and the motions resulting from these forces.**



# Kinematics

- The formulae that relates the above variables are as follows:

| | | |
|---|---|---|
| $v = at + v_0$ | [1] | $v = u + at$ |
| $r = r_0 + v_0 t + \frac{1}{2}at^2$ | [2] | $s = ut + \frac{1}{2}at^2$ |
| $r = r_0 + \frac{1}{2}(v + v_0)t$ | [3] | $s = \frac{1}{2}(u + v)t$ |
| $v^2 = v_0^2 + 2a(r - r_0)$ | [4] | $v^2 = u^2 + 2as$ |
| $r = r_0 + vt - \frac{1}{2}at^2$ | [5] | $s = vt - \frac{1}{2}at^2$ |

where:

- $r_0$ is the particle's initial position
- $r$ is the particle's final position
- $v_0$ is the particle's initial velocity
- $v$ is the particle's final velocity
- $a$ is the particle's acceleration
- $t$ is the time interval

Position, displacement, velocity, acceleration are *vectors*.

Time, speed are scalars.

# Kinematics

- The formulae that relates the above variables are as follows:

$$v = at + v_0 \qquad [1]$$

$$r = r_0 + v_0 t + \frac{1}{2} a t^2 \qquad [2]$$

$$r = r_0 + \frac{1}{2}(v + v_0) t \qquad [3]$$

$$v^2 = v_0^2 + 2a(r - r_0) \qquad [4]$$

$$r = r_0 + vt - \frac{1}{2} a t^2 \qquad [5]$$

- Above equations can be written in vector form as,

$$\mathbf{v} = \mathbf{a}t + \mathbf{v}_0 \qquad [1]$$

$$\mathbf{r} = \mathbf{r}_0 + \mathbf{v}_0 t + \frac{1}{2} \mathbf{a} t^2 \qquad [2]$$

$$\mathbf{r} = \mathbf{r}_0 + \frac{1}{2}(\mathbf{v} + \mathbf{v}_0) t \qquad [3]$$

$$\mathbf{v}^2 = \mathbf{v}_0^2 + 2\mathbf{a} \cdot (\mathbf{r} - \mathbf{r}_0) \qquad [4]$$

$$\mathbf{r} = \mathbf{r}_0 + \mathbf{v}t - \frac{1}{2} \mathbf{a} t^2 \qquad [5]$$

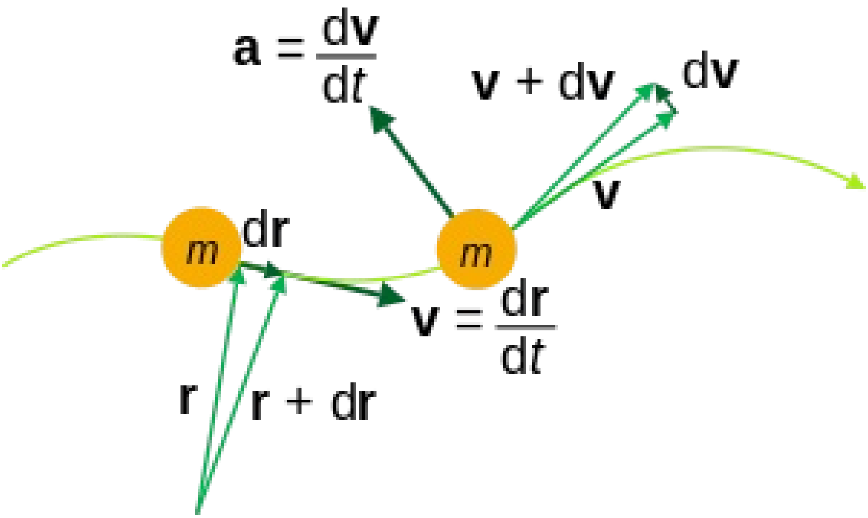
# Velocity & Acceleration - I

- The **position** of a particle is 3D is defined as $\vec{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$
- The **displacement** of a moving object is a vector connecting the initial to final position, $\vec{s} = \vec{r} - \vec{r}_0$
- The **velocity** of a moving particle at a time *t* is defined as, $\vec{v}(t) = \frac{d\vec{r}}{dt}$

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\,\hat{i} + \frac{dy}{dt}\,\hat{j} + \frac{dz}{dt}\,\hat{k}$$
$$= v_x\,\hat{i} + v_y\,\hat{j} + v_z\,\hat{k}$$

- The **acceleration** of a moving particle at a time *t* is defined as, $\vec{a}(t) = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = \frac{dv_x}{dt}\,\hat{i} + \frac{dv_y}{dt}\,\hat{j} + \frac{dv_z}{dt}\,\hat{k}$$
$$= \frac{d^2x}{dt^2}\,\hat{i} + \frac{d^2y}{dt^2}\,\hat{j} + \frac{d^2z}{dt^2}\,\hat{k}$$
$$= a_x\,\hat{i} + a_y\,\hat{j} + a_z\,\hat{k}$$





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# Example I

- Case I: A car is moving along a trajectory on map which can be expressed mathematically by $\vec{r}(t)=t^2\hat{i} + 2(t+1)\hat{j}$
Calculate the speed and acceleration of the car.

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = 2t\ \hat{i}+2\hat{j}$$

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = 2\hat{i} \quad .$$

- Case II: A car is accelerating at 2 m/s2 in the *x*- direction, $\vec{a}(t)= 2\hat{i}$
How is the position is changing in time and in which direction?

$$\vec{a}(t) = \frac{d\vec{v}}{dt} \Rightarrow d\vec{v} = \vec{a}\,dt \Rightarrow \int_{t_0}^{t} d\vec{v} = \int_{t_0}^{t} \vec{a}\ dt' \Rightarrow \vec{v}(t)-\vec{v}(t_0) = \int_{t_0}^{t} \vec{a}\ dt' \quad \text{or,} \quad \vec{v}(t) = \vec{v}(t_0)+\int_{t_0}^{t} \vec{a}\ dt'$$

$$\text{Again,} \quad \vec{v}(t) = \frac{d\vec{r}}{dt} \Rightarrow \int_{t_0}^{t} d\vec{r} = \int_{t_0}^{t} \vec{v}\ dt' \quad \text{or,} \quad \vec{r}(t) = \vec{r}(t_0)+\int_{t_0}^{t} \vec{v}\ dt'$$

- Two initial conditions are required to determine integration constants, ***r***($t_0$) and ***v***($t_0$).
Given that, initial position is ***r***($t_0$)=(0,2,0) and initial velocity is ***v***($t_0$)=(0,2,0) at $t_0$ = 0.

$$\text{Therefore,} \quad \vec{v}(t) = \vec{v}(t_0)+\int_{t_0}^{t} (2\hat{i})\ dt' = \vec{v}(t_0)+2(t-t_0)\hat{i} = 2\hat{j}+2t\hat{i}$$

$$\text{and,} \quad \vec{r}(t) = \vec{r}(t_0)+\int_{t_0}^{t} \vec{v}\ dt' = 2\hat{j}+\int_{t_0}^{t} (2t\hat{i}+2\hat{j})\ dt' = 2\hat{j}+(t^2-t_0^2)\hat{i}+2(t-t_0)\hat{j} = t^2\hat{i}+2(t+1)\hat{j}$$

# Condition on kinematic equations

- Kinematics equations: $\mathbf{v} = \mathbf{a}t + \mathbf{v}_0$ [1]

$$\mathbf{r} = \mathbf{r}_0 + \mathbf{v}_0 t + \frac{1}{2}\mathbf{a}t^2 \quad [2]$$

$$\mathbf{r} = \mathbf{r}_0 + \frac{1}{2}(\mathbf{v} + \mathbf{v}_0)\, t \quad [3]$$

$$\mathbf{v}^2 = \mathbf{v}_0^2 + 2\mathbf{a}\cdot(\mathbf{r} - \mathbf{r}_0) \quad [4]$$

$$\mathbf{r} = \mathbf{r}_0 + \mathbf{v}t - \frac{1}{2}\mathbf{a}t^2 \quad [5]$$

**are valid for constant acceleration and $t_0$=0.**

$$\vec{a}(t) = \frac{d\vec{v}}{dt} \quad \text{By integrating,} \quad \vec{v}(t) = \vec{v}(t_0) + \int_{t_0}^{t} \vec{a}\; dt' = \vec{v}_0 + \vec{a}\; t \quad \text{if } t_0 = 0 \text{ and } \vec{a} \text{ is constant (indep. of time)}.$$

$$\vec{v}(t) = \frac{d\vec{r}}{dt} \quad \text{By integrating,} \quad \vec{r}(t) = \vec{r}(t_0) + \int_{t_0}^{t} \vec{v}\; dt' = \vec{r}_0 + \int_{0}^{t} \vec{v}\; dt' = \vec{r}_0 + \int_{0}^{t} (v_0 + \vec{a}\, t')\; dt' = \vec{r}_0 + v_0 t + \frac{1}{2}\vec{a}\, t^2$$

# Example II

Electron in an oscillating electric field

- Imagine that an electron of charge $-e$ and mass $m$ is exposed to an electric field $\vec{E} = E_0 \sin \omega t\ \hat{\imath}$.
- Force $\vec{F}$ acting on the electron is $\vec{F} = -e\vec{E}$, so that $\vec{a}(t) = \frac{\vec{F}}{m} = -\frac{e}{m} E_0 \sin \omega t\ \hat{\imath}$
- As an initial conditions we consider at $t = 0$, electron is at rest, at the origin.
- Effectively, we have 1D motion in $x$ direction, so that $\frac{dv}{dt} = -\frac{e}{m} E_0 \sin \omega t \Rightarrow v(t) = v_0 - \frac{eE_0}{m} \int_0^t \sin \omega t'\, dt' = v_0 + \frac{eE_0}{m\omega} (\cos \omega t - 1)$.
- Since, $v_0 = 0$ thus $v(t) = \frac{eE_0}{m\omega} (\cos \omega t - 1)$.
- We obtain the trajectory by integrating the velocity equation, $x(t) = x_0 + \int_0^t v(t')\, dt' = x_0 + \frac{eE_0}{m\omega^2}(\sin \omega t - \omega t) = \frac{eE_0}{m\omega^2}(\sin \omega t - \omega t)$, as $x_0 = 0$.
- Note that besides an oscillating term, we also have a term which denotes drift of the electron with a constant velocity!

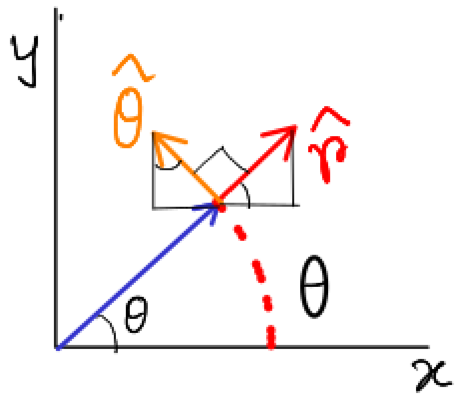
# Kinematics in plane polar coordinate - I

We can express velocity in plane polar coordinate as, $\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(r\hat{r}) = \frac{dr}{dt}\hat{r} + r\frac{d\hat{r}}{dt}$.

Note that, as the particle moves, vector $\hat{r}$ also changes so that $\frac{d\hat{r}}{dt} \neq 0$.

Hence, we need to compute $\frac{d\hat{r}}{dt}$.

$$\hat{r} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$$
$$\hat{\theta} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$$




Cartesian basis vectors are fixed in direction thus they are constant w.r.t time.

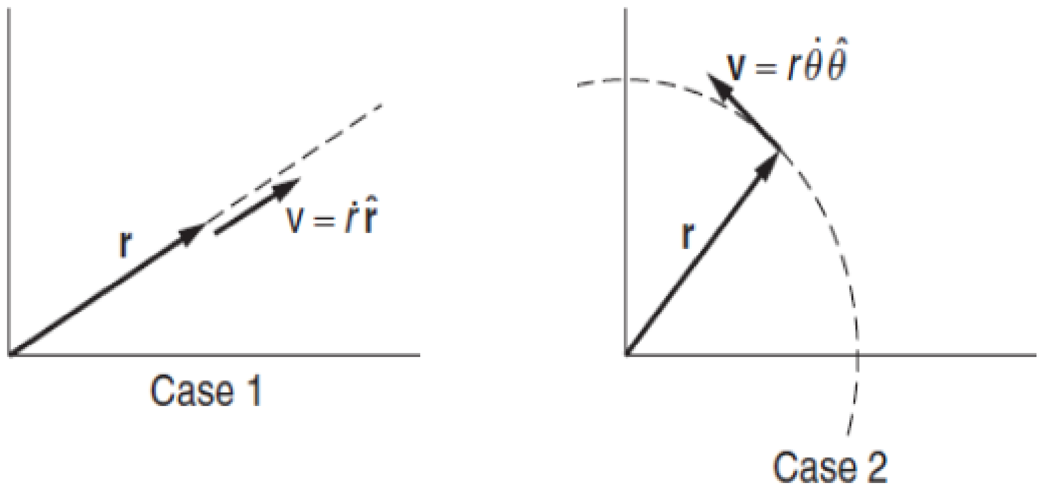
Hence, $\frac{d\hat{r}}{dt} = \frac{d\cos\theta}{dt}\hat{\imath} + \frac{d\sin\theta}{dt}\hat{\jmath} = -\sin\theta\,\dot{\theta}\hat{\imath} + \cos\theta\,\dot{\theta}\hat{\jmath} = \dot{\theta}(-\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}) = \dot{\theta}\hat{\theta}$

Thus, $\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} = v_r\hat{r} + v_\theta\hat{\theta}$

Thus, we have obtained an expression for velocity in terms of its radial and angular (also called tangential) components.

# Kinematics in plane polar coordinate - II

What is the physical significance of $v_r$ and $v_\theta$ ?




**Case 1:** This case corresponds to motion along the radial direction, with $\theta$ held fixed ($\dot{\theta} = 0$), so that $\vec{v} = \dot{r}\hat{r}$

**Case 2:** Here there is no radial motion ($\dot{r} = 0$), so velocity will be along the arc of a circle with $\vec{v} = r\dot{\theta}\hat{\theta}$.

# Kinematics in plane polar coordinate - III

❖Acceleration can be computed as, $\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta}\right) = \frac{d\dot{r}}{dt}\hat{r} + \dot{r}\frac{d\hat{r}}{dt} + \frac{d}{dt}\left(r\dot{\theta}\right)\hat{\theta} + r\dot{\theta}\frac{d\hat{\theta}}{dt}$.

❖This results, $\vec{a} = \ddot{r}\hat{r} + \dot{r}\left(\dot{\theta}\hat{\theta}\right) + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}\frac{d\hat{\theta}}{dt}$.

❖Since, $\hat{\theta} = -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}$, thus $\frac{d\hat{\theta}}{dt} = \frac{d}{dt}(-\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}) = -\cos\theta\,\dot{\theta}\hat{\imath} - \sin\theta\,\dot{\theta}\hat{\jmath} = -\dot{\theta}\hat{r}$.

❖Hence, $\vec{a} = \ddot{r}\hat{r} + \dot{r}\dot{\theta}\hat{\theta} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}\left(-\dot{\theta}\hat{r}\right) = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{r} + \left(\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{\theta}$

✓$\ddot{r}$ is due to change of radial speed, points in radial direction

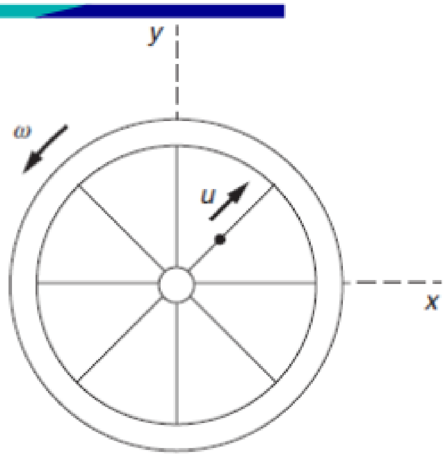
✓$-r\dot{\theta}^2$ is the centripetal acceleration, pointing radially inwards

✓$2\dot{r}\dot{\theta}$ describes the Coriolis acceleration, present whenever both radial and angular velocities are nonzero, points in tangential direction.

✓$r\ddot{\theta}$ is the tangential angular acceleration, due to changing angular velocity, points in tangential direction.

# Example

Consider a bead moving along the spoke of a rotating wheel. Assume both $u$ and $\omega$ are constant. Calculate the velocity and acceleration of the bead in plane polar coordinates.




Here, $\dot{r} = u; \dot{\theta} = \omega; \ddot{r} = 0; \ddot{\theta} = 0.$

Thus, velocity in polar coordinate is $\vec{v} = u\hat{r} + r\omega\hat{\theta}.$

However, in this case $r = ut$, hence, $\vec{v} = u\hat{r} + ut\omega\hat{\theta}.$

The acceleration can now be computed as, $\vec{a} = -\omega^2 r\hat{r} + 2u\omega\hat{\theta} = -\omega^2 ut\hat{r} + 2u\omega\hat{\theta}$

$$\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

$$\vec{a} = \ddot{r}\hat{r} + \dot{r}\dot{\theta}\hat{\theta} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}(-\dot{\theta}\hat{r}) = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}$$