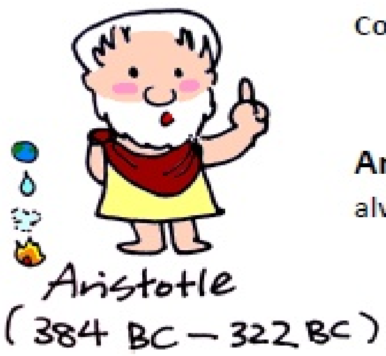
Kinetics

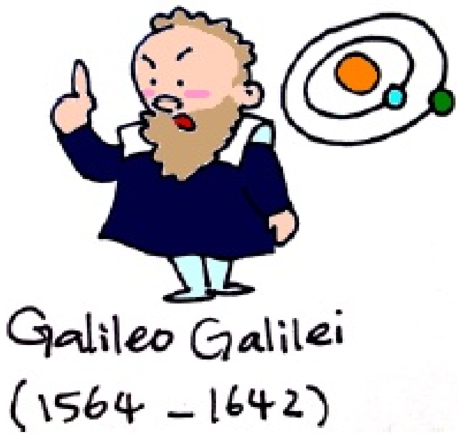


Consider a rolling ball...

Aristotle thought that an object needs a force to keep it moving. A rolling ball always comes to a stop. It needs a force to keep it moving.

Aristotle **VS** Galileo

Galileo thought that a force is not needed to keep an object moving. He argued that the ball stops because it is acted upon by friction between the ball and the surface. If there is no friction, the ball will continue to move forever.



Whom do you agree with?

Aristotle Like | Galileo Like

Many agreed with Aristotle that time but one scientist **Isaac Newton** agreed with Galileo. He developed Galileo's thought on motion and came up with his first law of motion.

Newton's first law of motion states that :

An object will remain stationary or continue in uniform motion in a straight line unless acted upon by an external force.

Newton’s laws of motion

- **Newton’s 1st law:** Every body continues in its state of rest, or of uniform motion in a straight line, unless it is compelled to change that state by forces impressed upon it.

  (principle of intertia)

- **Newton’s 2nd law:** The change of motion of an object is proportional to the force applied; and is made in the direction of the straight line in which the force is applied.

  (Quantification of force and motion)

- **Newton’s 3rd law:** To every action, there is always an oppsite and equal reaction. (conservation of momentum)

Newton's 2nd law: Force & momentum

- Let us first define the momentum ($\vec{p}$) of a particle as $\vec{p} \;=\; m\vec{v},$ where m is the mass of the system, and v is its velocity.
- Newtons' second law of motion states that force $\vec{F}$ acting on a system is equal to its rate of change of momentum or $\vec{F} = \frac{d\vec{p}}{dt}$
- This is the most general definition of Newton's second law of motion and is applicable also to those systems, such as a rocket, whose mass m is not constant.
- However, for a system whose mass does not change with time, we have its more familiar form $\vec{F} = \frac{d}{dt}(m\vec{v}) = m\frac{d\vec{v}}{dt} = m\vec{a}$

Work & energy (1D)

- Consider a force ***F****(x)* is acting on a particle of mass *m* that is moving in 1D. Note the force varies with position. *A* position dependent vector quantity like force is called a "vector field".
- The work done in moving the particle by an infinitesimal amount *d**x*** is dW = ***F****(x).d**x***
- The work done in moving the particle from *a* to *b* will be

$$W_{ab} = \int_a^b F(x)dx = \int_a^b ma\, dx = m\int_a^b \frac{dv}{dt}\, dx.$$

Now, $dx = \frac{dx}{dt}\, dt = v\, dt$

Therefore, $W_{ab} = m\int_a^b \frac{dv}{dt}\, vdt = \int_a^b \frac{d}{dt}\left(\frac{1}{2}\, mv^2\right)dt = \frac{1}{2}\, mv_b^2 - \frac{1}{2}m\, v_a^2$

In 1D, work done = chang in kinetic energy. What about in 3D?

- Consider a force ***F(r)*** is acting on a particle of mass *m* that is moving in 3D.

The work done in displacing the particle by a finite amount, starting from $\vec{r} = \vec{r}_a$ to $\vec{r} = \vec{r}_b$ will be $W_{ab} = \int_{\vec{r}_a}^{\vec{r}_b} \vec{F}\cdot d\vec{r}$

$$W_{ab} = \int_{\vec{r}_a}^{\vec{r}_b} \vec{F}\cdot d\vec{r} = \int_{\vec{r}_a}^{\vec{r}_b} m\frac{d\vec{v}}{dt}\cdot\vec{v}\, dt = \int_{\vec{r}_a}^{\vec{r}_b} \frac{d}{dt}\left(\frac{1}{2}m\,\vec{v}\cdot\vec{v}\right)dt = \int_{\vec{r}_a}^{\vec{r}_b} \frac{d}{dt}\left(\frac{1}{2}mv^2\right)dt = \frac{1}{2}mv_b^2 - \frac{1}{2}mv_a^2$$

Work & energy (3D)

- Consider a force ***F(r)*** is acting on a particle of mass *m* that is moving in 3D.

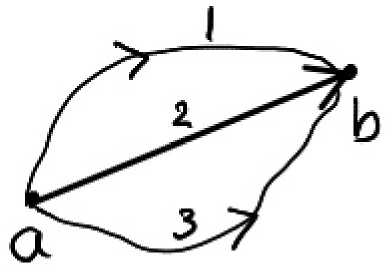
  The work done in displacing the particle by a finite amount, starting from $\vec{r} = \vec{r}_a$ to $\vec{r} = \vec{r}_b$ will be $W_{ab} = \int_{\vec{r}_a}^{\vec{r}_b} \vec{F} \cdot d\vec{r}$

  Now, $d\vec{r} = \frac{d\vec{r}}{dt}\,dt = \vec{v}\,dt$

$$W_{ab} = \int_{\vec{r}_a}^{\vec{r}_b} \vec{F} \cdot d\vec{r} = \int_{\vec{r}_a}^{\vec{r}_b} m\frac{d\vec{v}}{dt} \cdot \vec{v}\,dt = \int_{\vec{r}_a}^{\vec{r}_b} \frac{d}{dt}\left(\frac{1}{2}m\,\vec{v}\cdot\vec{v}\right)dt =$$

$$\int_{\vec{r}_a}^{\vec{r}_b} \frac{d}{dt}\left(\frac{1}{2}mv^2\right)dt = \frac{1}{2}mv_b^2 - \frac{1}{2}mv_a^2$$

- Work done is expressed in terms of a line integral, so in principle it must depend on tha path connecting points a and b.
  For example, for the three paths shown in the figure,
  the line integral, in general, will have three different values



- Do we have forces F(r) for which this line integral is path independent?

Conservative force

- For most fundamental forces in nature the work done does not depend on the path of displacement. Examples: gravitational force, electrostatic force

  Rather it will depend only on the positions of the end points (a and b in this case) of the path

  Such forces are called “Conservative Forces”

- For conservative forces, there exists a mathematical function V(***r***) such that

$$\int_{\vec{r}_a}^{\vec{r}_b} \vec{F} \cdot d\vec{r} = -(V(\vec{r}_b) - V(\vec{r}_a))$$

  If for a forece, we can find such a function that work done can be expressed as in RHS, then the force is conservative force.
  -ve sign at RHS is matter of convention.

- V(r) has dimention if energy and scalar.
- It is obvious that for a conservative force, work done along a closed path is zero.

$$\oint \vec{F} \cdot d\vec{r} = \int_{\vec{r}_a}^{\vec{r}_b} \vec{F} \cdot d\vec{r} + \int_{\vec{r}_b}^{\vec{r}_a} \vec{F} \cdot d\vec{r} = -\left(V(\vec{r}_b) - V(\vec{r}_a)\right) - \left(V(\vec{r}_a) - V(\vec{r}_b)\right) = 0$$

Potential energy I

- A consequesnce of the work-energy theorem for conservative force is that sum of kinetic and potential energies of a system is conserved.
  (..hence the name “conservative” force)

  For conservative force, we have

  $$W_{ab} = \int_{\vec{r}_a}^{\vec{r}_b} \vec{F} \cdot d\vec{r} = \frac{1}{2}mv_b^2 - \frac{1}{2}mv_a^2 = V(\vec{r}_a) - V(\vec{r}_b)$$

  This suggests,

  $$\frac{1}{2}mv_a^2 + V(\vec{r}_a) = \frac{1}{2}mv_b^2 + V(\vec{r}_b)$$

- How do we define the potential energy *V(**r**)* at a given point ***r*** in space?

  Potential energy is work done in bringing a point from infinity to the point ***r*** against the force ***F*** *.*

  $$V(\vec{r}) = -\int_{\vec{r}_O}^{\vec{r}} \vec{F}(\vec{r}) \cdot d\vec{r}$$

  For 1D conservative force, $V(b) - V(a) = -\int_a^b F(x)\,dx$

Potential energy II

- How to find *V(**r**)?* $\int_{\vec{r}_a}^{\vec{r}_b} \vec{F} \cdot d\vec{r} = -(V(\vec{r}_b) - V(\vec{r}_a))$

  Let's take $\vec{r}_a = \vec{r}$ and $\vec{r}_b = \vec{r} + \Delta\vec{r}$ where $\Delta\vec{r}$ is an infinitesimal displacement vector in 3D.

  This leads to $V(\vec{r} + \Delta\vec{r}) - V(\vec{r}) = -\int_{\vec{r}}^{\vec{r}+\Delta\vec{r}} \vec{F} \cdot d\vec{r}'$

  Now, RHS: $-\int_{\vec{r}}^{\vec{r}+\Delta\vec{r}} \vec{F} \cdot d\vec{r}' \sim -\vec{F}(\vec{r}) \cdot \Delta\vec{r} = -F_x\Delta x - F_y\Delta y - F_z\Delta z$

  Expanding LHS in Taylor series:

  $$V(\vec{r} + \Delta\vec{r}) = V(x + \Delta x, y + \Delta y, z + \Delta z) = V(x, y, z) + \frac{\partial V}{\partial x}\Delta x + \frac{\partial V}{\partial y}\Delta y + \frac{\partial V}{\partial z}\Delta z$$

  $$= V(\vec{r}) + \vec{\nabla}V \cdot \Delta\vec{r}$$

  Hence,

  $$V(\vec{r} + \Delta\vec{r}) - V(\vec{r}) = \nabla V \cdot \Delta\vec{r} = -\vec{F}(\vec{r}) \cdot \Delta\vec{r} \Rightarrow \vec{F} = -\vec{\nabla}V$$

- This is very important result showing that a conservative force can be written as the gradient of corresponding potential energy.

Example

- A conservative force is given by a vector: $\vec{F} = A(x^2\hat{\imath} + y\hat{\jmath})$
  Find V(r).

$$-\vec{\nabla}V = \vec{F}$$

$$\frac{\partial V}{\partial x} = -Ax^2 \text{ and } \frac{\partial V}{\partial y} = -Ay$$

On integrating the first equation we obtain, $V(x,y) = -\frac{Ax^3}{3} + f(y)$ where $F(y)$ is an unknown function of y. Substituting this result in the second equation we have,

$$\frac{\partial}{\partial y}\left(-\frac{Ax^3}{3} + f(y)\right) = -Ay \Rightarrow \frac{\partial f}{\partial y} = -Ay \Rightarrow f(y) = -\frac{Ay^2}{2} + C$$

Thus, $V(x,y) = -\frac{A}{3}x^3 - \frac{A}{2}y^2 + C$