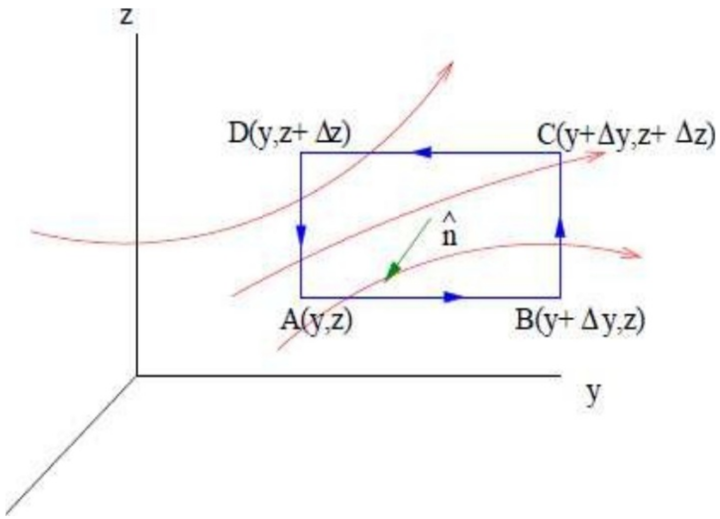
Line integral of vector field

- We learnt in kinetics that for a conservative force $\vec{F}$:

1. $W_{ab} = \int_{\vec{r}_a}^{\vec{r}_b} \vec{F} \cdot d\vec{r} = \frac{1}{2} m v_b^2 - \frac{1}{2} m v_a^2 = V(\vec{r}_a) - V(\vec{r}_b)$
2. $\frac{1}{2} m v_a^2 + V(\vec{r}_a) = \frac{1}{2} m v_b^2 + V(\vec{r}_b)$
3. $V(\vec{r}) = -\int_{\vec{r}_O}^{\vec{r}} \vec{F}(\vec{r}) \cdot d\vec{r}$
4. $\vec{F} = -\vec{\nabla} V$
5. $\oint \vec{F} \cdot d\vec{r} = 0$



- Let us study the line integral of force a little more.

Consider a vector field $\vec{F} = F_x \hat{\imath} + F_y \hat{\jmath} + F_z \hat{k}$ and evaluate the line integral along infinitesimal rectangular path as shown in figure.

$$\oint \vec{F} \cdot d\vec{l} = \int_{AB} \vec{F} \cdot d\vec{l} + \int_{BC} \vec{F} \cdot d\vec{l} + \int_{CD} \vec{F} \cdot d\vec{l} + \int_{DA} \vec{F} \cdot d\vec{l}$$

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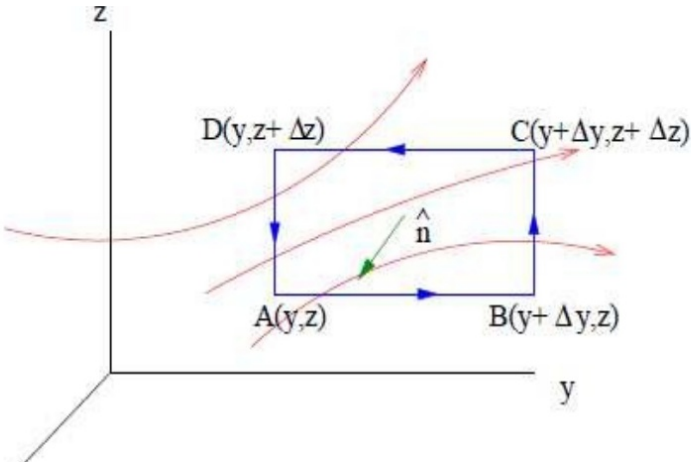


- Now according to the figure

$$\int_{AB} \vec{F} \cdot d\vec{l} = \int \vec{F} \cdot dy\, \hat{j} \sim F_y(y,z)\Delta y$$

$$\int_{BC} \vec{F} \cdot d\vec{l} = \int \vec{F} \cdot dz\, \hat{k} \sim F_z(y+\Delta y, z)\Delta z$$

- Using Taylor expansion:

$$F_z(y+\Delta y, z) = F_z(y,z) + \frac{\partial F_z}{\partial y}\,\Delta y$$

$$\int_{AB} \vec{F} \cdot d\vec{l} + \int_{BC} \vec{F} \cdot d\vec{l} = \left(F_y \Delta y + F_z \Delta z + \frac{\partial F_z}{\partial y}\,\Delta y \Delta z\right)$$

- Similarly, one can show for AD and DC directions.

$$\int_{CD} \vec{F} \cdot d\vec{l} + \int_{DA} \vec{F} \cdot d\vec{l} = \left(F_y \Delta y + F_z \Delta z + \frac{\partial F_y}{\partial z}\,\Delta y \Delta z\right)$$

- If we add all the contributions, we get

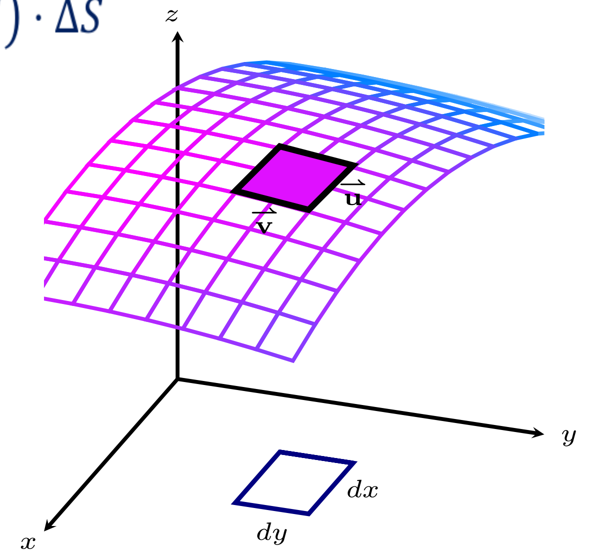
$$\oint \vec{F} \cdot d\vec{l} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\Delta S_x$$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

$$\text{Hence, } \oint \vec{F} \cdot d\vec{l} = (\vec{\nabla} \times \vec{F})_x \Delta S_x = (\vec{\nabla} \times \vec{F}) \cdot \Delta\vec{S}$$

Stoke's theorem

- If we take a closed loop enclosing a finite area, we can divide the whole area into many infinitesimal area.
- For each infinitesimal area, holds $\oint \vec{F} \cdot d\vec{l} = (\vec{\nabla} \times \vec{F})_x \Delta S_x = (\vec{\nabla} \times \vec{F}) \cdot \Delta\vec{S}$
- Add up the contribution of all infinitesimal areas to get the line integral of the total finite area,



$$\oint \vec{F}.\vec{dl} = \iint_S \vec{\nabla} \times \vec{F}.d\vec{S}$$

Note that in the line integral, the contribution only from the boundary of the loop will survive because the contribution from the internal lines gets cancelled from adjacent loops.

- **Stoke's theorem:**

If a vector field $\vec{F}$ is integrated along a closed loop of an arbitrary shape, then the line integral is equal to the surface integral of the curl of $\vec{F}$ evaluated across the area enclosed by the loop.

$$\oint \vec{F}.\vec{dl} = \iint_S \vec{\nabla} \times \vec{F}.d\vec{S}$$

Curl & conservative force

- If $\vec{F}$ is a conservative force, $\oint \vec{F}.\vec{dl} = 0$
- Then from Stoke's theorem: $\oint \vec{F}.\vec{dl} = \iint_S \vec{\nabla} \times \vec{F}.d\vec{S} = 0$

  But the surface area enclosed by a closed loop is in general zero. Hence, $\vec{\nabla} \times \vec{F} = 0$

  Thus, ***all conservative forces have vanishing curl.***

- Moreover, for conservative force, $\vec{F} = -\vec{\nabla} V(\vec{r}) = -\frac{\partial V}{\partial x}\hat{\imath} - \frac{\partial V}{\partial y}\hat{\jmath} - \frac{\partial V}{\partial z}\hat{k}$

  We know curl of gradient of any scalar function vanishes, $-\vec{\nabla} \times \vec{\nabla} V = 0$

$$\text{Now, } \vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\frac{\partial V}{\partial x} & -\frac{\partial V}{\partial y} & -\frac{\partial V}{\partial z} \end{vmatrix}$$

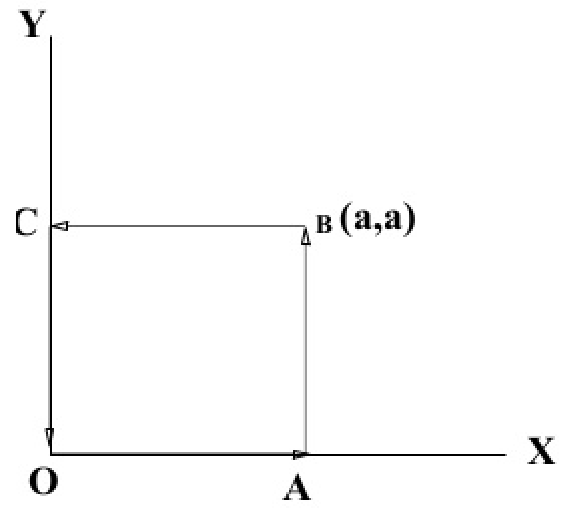
$$= \left(-\frac{\partial^2 V}{\partial y \partial z} + \frac{\partial^2 V}{\partial z \partial y}\right)\hat{\imath} + \left(-\frac{\partial^2 V}{\partial z \partial x} + \frac{\partial^2 V}{\partial x \partial z}\right)\hat{\jmath} + \left(-\frac{\partial^2 V}{\partial x \partial y} + \frac{\partial^2 V}{\partial y \partial x}\right)\hat{k} = 0$$

iff, $\frac{\partial^2 V}{\partial y \partial z} = \frac{\partial^2 V}{\partial z \partial y}$ and so on.

Example

- Let's understand Stoke's theorem through an example.
  Consider a 2D vector field $\vec{F} = -y\,\hat{i} + x\,\hat{j}$. Define a closed loop as shown in figure.

- Calculation of LHS of Stoke's theorem:



$$\oint \vec{F}\cdot d\vec{l} = \int_{OA} \vec{F}\cdot d\vec{l} + \int_{AB} \vec{F}\cdot d\vec{l} + \int_{BC} \vec{F}\cdot d\vec{l} + \int_{CO} \vec{F}\cdot d\vec{l}$$

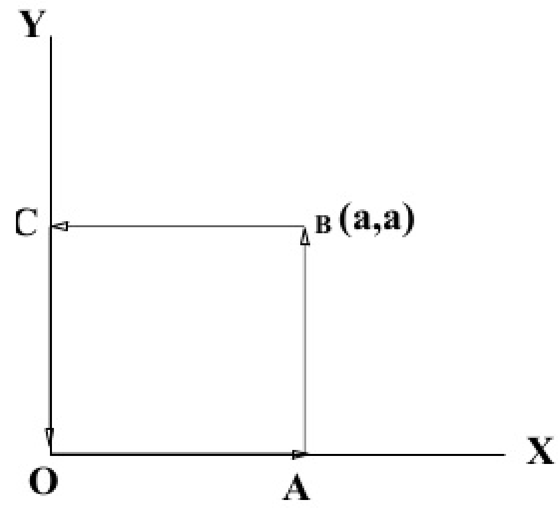
- Now,
  - $\int_{OA} \vec{F}\cdot d\vec{l} = \int \vec{F}\cdot dx\,\hat{\imath} = \int_0^a (-y)dx = 0$ as $y = 0$.
  - $\int_{AB} \vec{F}\cdot d\vec{l} = \int \vec{F}\cdot dy\,\hat{\jmath} = \int_0^a xdy = a^2$.
  - $\int_{BC} \vec{F}\cdot d\vec{l} = \int \vec{F}\cdot dx\,\hat{\imath} = \int_a^0 (-y)dx = a^2$.
  - $\int_{CO} \vec{F}\cdot d\vec{l} = \int \vec{F}\cdot dy\,\hat{\jmath} = \int_a^0 x\,dy = 0$ as $x = 0$.

$$\oint \vec{F}\cdot d\vec{l} = a^2 + a^2 = 2a^2$$

Example

- Let's understand Stoke's theorem through an example.
  Consider a 2D vector field $\vec{F} = -y\,\hat{i} + x\,\hat{j}$. Define a closed loop as shown in figure.

- Now, let us calculate RHS, $\int_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{S}$.



$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix} = 2\hat{k}$$

$$d\vec{S} = dxdy\hat{k}$$

Hence, $\int_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{S} = 2\int dx \int dy = 2a^2$

This clearly shows that LHS=RHS, therefore Stokes' theorem is verified.

Example

- A conservative force is given by a vector: $\vec{F} = A(x^2\hat{\imath} + y\hat{\jmath})$
  Find V(r).

$$-\vec{\nabla}V = \vec{F}$$

$$\frac{\partial V}{\partial x} = -Ax^2 \text{ and } \frac{\partial V}{\partial y} = -Ay$$

On integrating the first equation we obtain, $V(x,y) = -\frac{Ax^3}{3} + f(y)$ where $F(y)$ is an unknown function of y. Substituting this result in the second equation we have,

$$\frac{\partial}{\partial y}\left(-\frac{Ax^3}{3} + f(y)\right) = -Ay \Rightarrow \frac{\partial f}{\partial y} = -Ay \Rightarrow f(y) = -\frac{Ay^2}{2} + C$$

Thus, $V(x,y) = -\frac{A}{3}x^3 - \frac{A}{2}y^2 + C$

V(r) can be found in this method if $-\vec{\nabla}V = \vec{F}$, that is if ***F*** is a conservative force.

So we must first check if $\vec{\nabla} \times \vec{F} = 0$