Harmonic oscillator

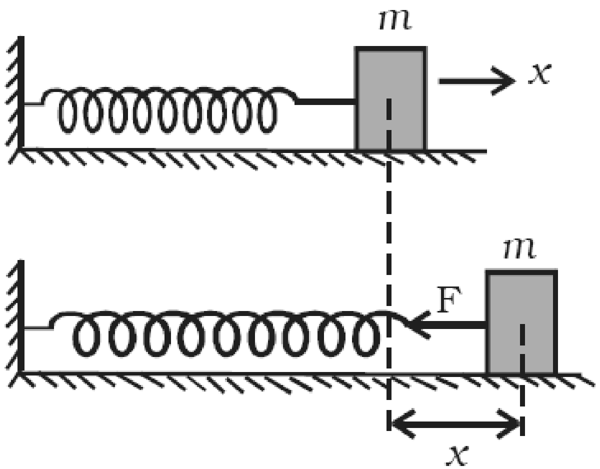
- Immensely important topic, studied in Classical mechanics, Quantum mechanics with many applications.
- A classical harmonic oscillator (in 1D) is described by a mass *m* attached to a spring of spring constant *k.* The motion of the mass is governed by Hooke's law:

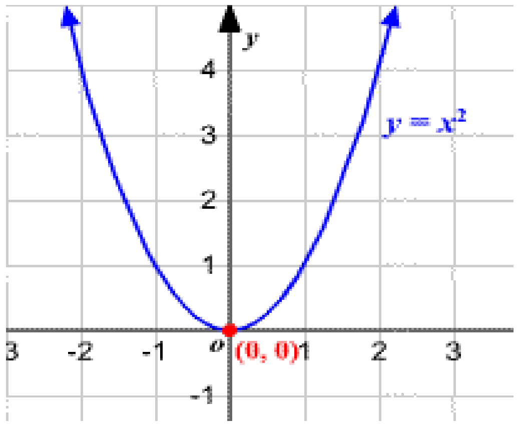
$$F = -kx = m\frac{d^2x}{dt^2}$$

where *x* is extension of the spring from the unstretched position, *F* is the restoring force.

- As force can be written as *F* = - *dV/dx,* the potential energy becomes parabolic:

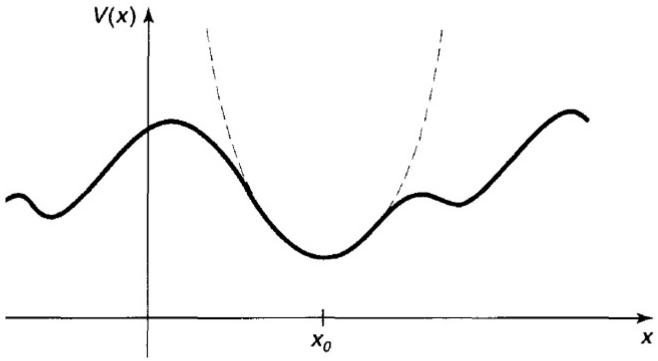
$$V(x) = \frac{1}{2}kx^2$$





Why important?

- Any oscillatory motion can be approximated as a simple harmonic motion as long as the amplitude of oscillation is small.
  Practically any potential is approximately parabolic around the local minimum.



- If we expand an arbitrary function around its minimum, in a Taylor series,

$$V(x) = V(x_0) + V'(x_0)(x - x_0) + \frac{1}{2}V''(x_0)(x - x_0)^2 + \cdots$$

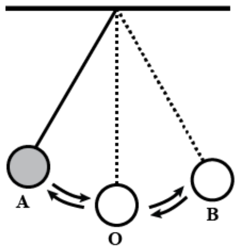
- Substracting $V(x_0)$ from both sides (it does not affect force as force is derivative of $V(x)$) and noting that at the minimum x_0 , $V'(x_0) = 0$, we get

$$V(x) \cong \frac{1}{2}V''(x_0)(x - x_0)^2$$

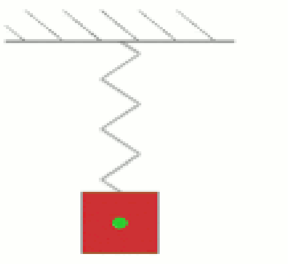
Periodic / Oscillatory / Simple harmonic motion

- **Periodic motion:** Motion which repeats itself after a regular interval. In periodic motion displacement of the object may not be in the direction of restoring force.
  Example: Earth's rotation around sun, Hands of clock, Pendulum of a clock.

- **Oscillatory motion:** Periodic motion on same path, to and fro.
  A *periodic motion* may or may not be *oscillatory*.
  Example: Pendulum of clock



- **Simple Harmonic Motion (SHM):** a linear oscillatory motion where displacement of the object is *always* in the opposite direction of the restroring force, and restoring force is proportional to the distace from the equilibrium position x_0 .
  A small amplitude oscillatory motion can be considered as SHM.



Simple Harmonic Motion - I

- A classical harmonic oscillator (in 1D) is described by a mass *m* attached to a spring of spring constant *k.* The motion of the mass is governed by:

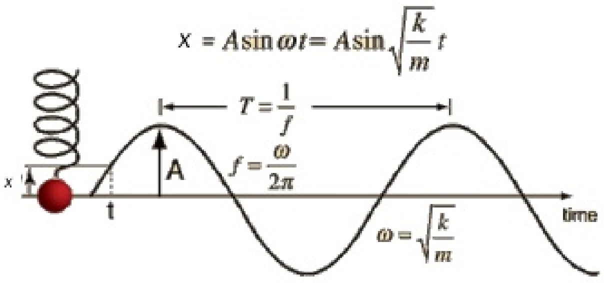
$$F = -kx = m\frac{d^2x}{dt^2}$$

(we have neglected friction!)

- Solution of the differential equation describes the motion of the SHM:

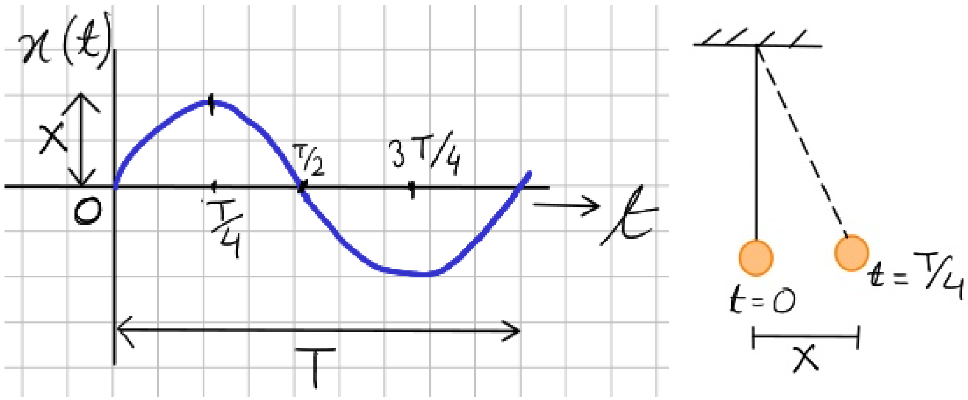
$$m\frac{d^2x}{dt^2} + kx = 0$$

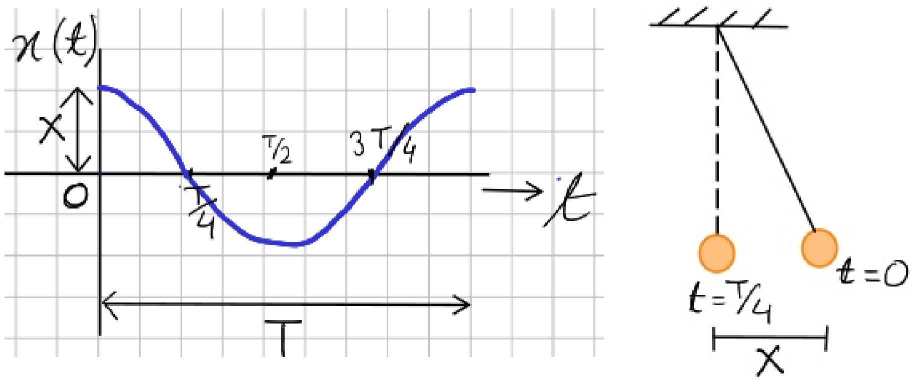
$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$



$$x(t) = A\sin\omega t + B\cos\omega t$$

A = ? B = ?





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Simple Harmonic Motion - II

In a more convenient form $x(t) = C\cos(\omega t + \phi)$, where C and ϕ are constants.

Comparing with the previous solution we can write, $C\sin\phi = -A$ and $C\cos\phi = B$.

Thus, $\tan\phi = -\frac{A}{B}$

Potential Energy (P): $\frac{1}{2}kx^2 = \frac{1}{2}kC^2\cos^2(\omega t + \phi)$

Kinetic Energy (K): $\frac{1}{2}mv^2 = \frac{1}{2}m\omega^2C^2\sin^2(\omega t + \phi)$

Hence, **Total energy (E):** K+P=$\frac{1}{2}kC^2[\sin^2(\omega t + \phi) + \cos^2(\omega t + \phi)] = \frac{1}{2}kC^2$

This implies *total energy is constant* as the applied force is conservative.

Example

If a 4.0-kg mass is suspended from the lower end of a coiled spring, it stretches a distance of 18.0cm. If the spring is then extended farther and released, it will be set vibrating up and down with SHM. Find *(a)* the spring constant *k, (b)* the period *T, (c)* the frequency v, and (d) the *total energy* stored in the vibrating system.

The spring constant $$k = -\frac{F}{x} = -\frac{mg}{x} = -\frac{4\times 9.8}{0.18} = 217.8N/m$$

The period $$T = 2\pi\sqrt{\frac{m}{k}} = 0.852s$$

The frequency $$\nu = \frac{1}{T} = 1.174Hz$$

The total energy $$E = \frac{1}{2}kx^2 = 3.528Nm = 3.528J$$