Damped Harmonic Oscillator - I

- Simple harmonic oscillator is an ideal case where there is frictional or opposing force. This can be achieved with ideally massless spring, very heavy bob, frictionless motion.
- The damping in harmonic oscillator (HO) happens due to friction.

  We will consider here, a viscous drag affecting the damping.

  Such that, $\vec{f} = -b\vec{v}$. Here, $\vec{f}$ is the frictional force.

  b is a proportionality constant that depends on the shape of the mass and the medium through which it moves.

  $\vec{v}$ is the instantaneous velocity.

  Hence, total force $\vec{F} = \vec{F}_{spring} + \vec{f} = -k\vec{x} - b\vec{v}$.

  The equation of motion: $m\ddot{x} = -kx - b\dot{x} \Rightarrow \ddot{x} + \gamma\dot{x} + \omega^2 x = 0$

  Here, $\gamma = b/m$ and $\omega^2 = k/m$

Solving Diff. Eqn. for dampled H.O.

The exponential ansatz states that a special solution to the differential equation takes the form:

$$x(t) = Ce^{\lambda t}$$

By forming the first and second derivations of (2) we obtain:

$$\dot{x}(t) = \lambda Ce^{\lambda t} \ddot{x}(t) = \lambda^2 Ce^{\lambda t}$$

By inserting the equations (2) and (3) into the differential equation (1) we obtain:

$$\lambda^2 Ce^{\lambda t} + \frac{\mu}{m}\lambda Ce^{\lambda t} + \frac{k}{m}Ce^{\lambda t} = 0$$

Division by $Ce^{\lambda t}$ will simplify this to:

$$\lambda^2 + \frac{\mu}{m}\lambda + \frac{k}{m} = 0$$

This equation is also called the characteristic equation. It is a quadratic equation in normal form.

Damped Harmonic Oscillator - II

- Hence, the *characteristic equation* can be written as: $\lambda^2 + \gamma\lambda + \omega^2 = 0$

$$\Rightarrow \lambda = \frac{-\gamma \pm \sqrt{\gamma^2 - 4\omega^2}}{2}$$

- The *particular solution* for the equation can be noted as $x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$

- One can write a general solution such as: $x(t) = Ae^{-\left(\frac{\gamma}{2}\right)t} \cos(\omega_1 t + \phi)$, where $\omega_1 = \sqrt{\omega^2 - \frac{\gamma^2}{4}}$ and ϕ is the phase.

Light Damping: $\gamma^2 \ll 4\omega^2$

- In this case, $\gamma^2 \ll 4\omega^2$, $x(t) = C_1 e^{-\frac{\gamma}{2}t + i\sqrt{\omega^2 - \frac{\gamma^2}{4}}} + C_2 e^{-\frac{\gamma}{2}t - i\sqrt{\omega^2 - \frac{\gamma^2}{4}}}$
- So, the solution will be oscillatory in nature, but the amplitude will be decaying exponentially.

Damped Harmonic Oscillator - III

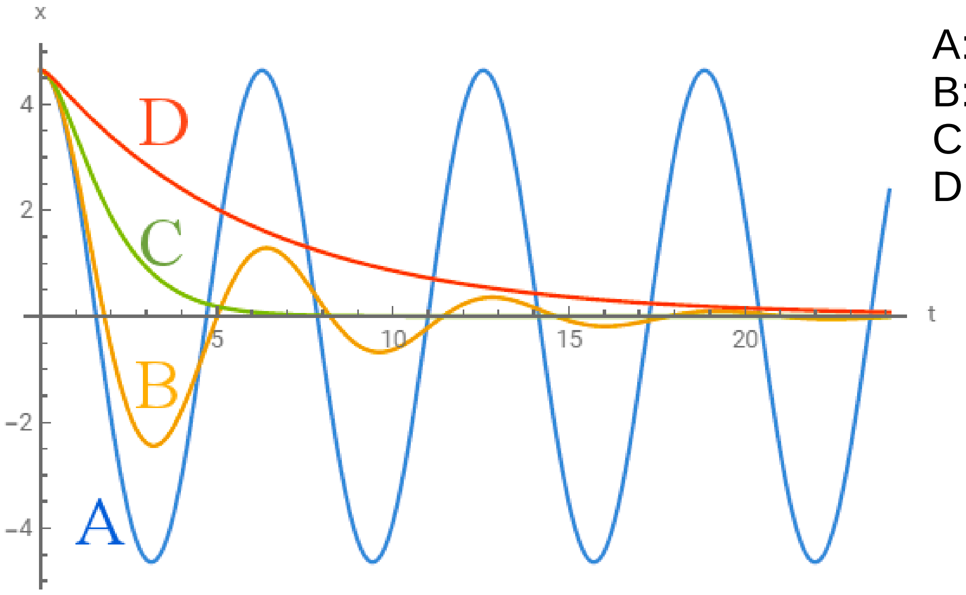
Heavy Damping: $\gamma^2 > 4\omega^2$

- In this case, $x(t) = C_1 e^{\left(-\frac{\gamma}{2}+\frac{\gamma}{2}\sqrt{1-\frac{4\omega^2}{\gamma^2}}\right)t} + C_2 e^{\left(-\frac{\gamma}{2}-\frac{\gamma}{2}\sqrt{1-\frac{4\omega^2}{\gamma^2}}\right)t}$
- The solution does not have any oscillatory behavior. The motion is known as overdamped.

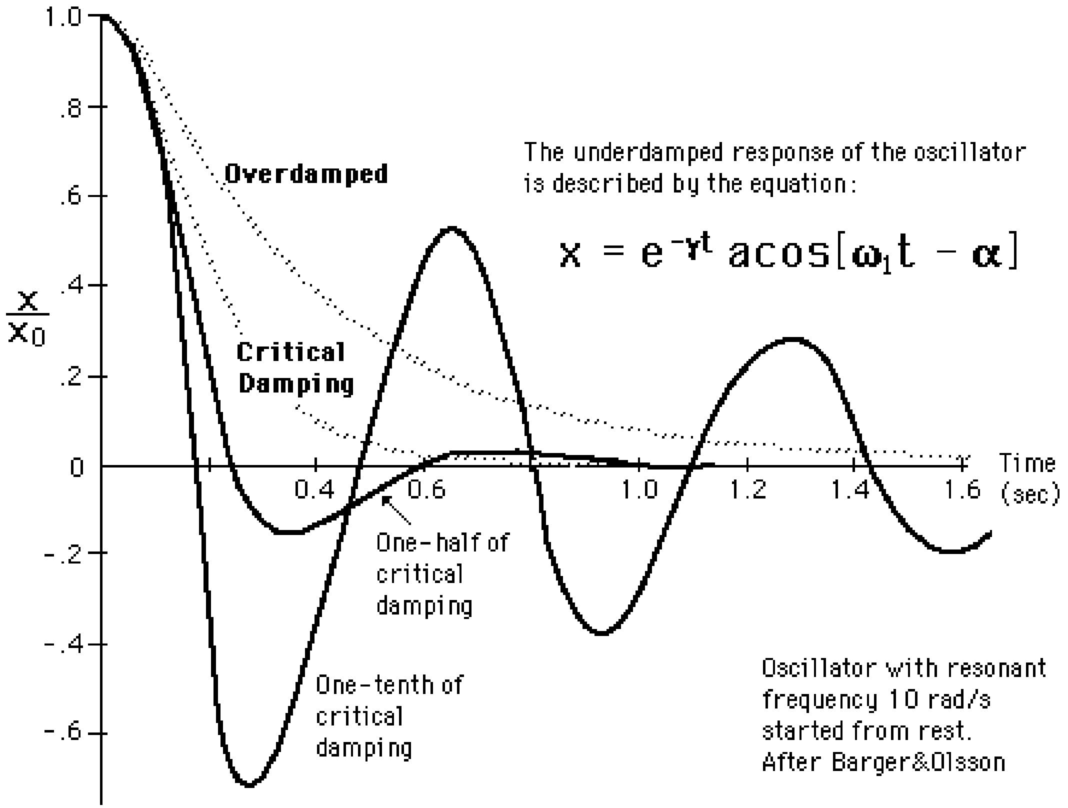
Critical Damping: $\gamma^2 = 4\omega^2$

- In this case, $x(t) = Ce^{-\frac{\gamma t}{2}}$
- However, the solution is incomplete as a second order linear differential equation must contain two arbitrary constant. Hence, we consider $x(t) = u(t)e^{-\frac{\gamma}{2}t}$.
- Substituting in the original equation, and using the fact $\gamma = 2\omega$ we find that $u(t)$ must satisfy $\ddot{u} = 0$.
- Hence, $u(t) = a + bt$
- So, the general solution will be: $x(t) = Ae^{-\frac{\gamma}{2}t} + Bte^{-\frac{\gamma}{2}t}$

Comparison



A: Undamped
B: Under-damped
C: Critically damped
D: Over-damped



https://www.youtube.com/watch?v=99ZE2RGwqSM

https://www.youtube.com/watch?v=sP1DzhT8Vzo

https://beltoforion.de/en/harmonic_oscillator/

Lectures by Walter Lewin:

https://www.youtube.com/watch?v=tNpuTx7UQbw
https://www.youtube.com/watch?v=77ZF50ve6rs

Damped Harmonic Oscillator - IV

- If we consider the oscillator is lightly damped, then we can assume
- $x(t) = Ae^{-\left(\frac{\gamma}{2}\right)t}\cos(\omega_1 t + \phi) \Rightarrow v(t) = -Ae^{-\frac{\gamma}{2}t}\left[\frac{\gamma}{2}\cos(\omega_1 t + \phi) + \omega_1 \sin(\omega_1 t + \phi)\right] = -\omega_1 Ae^{-\frac{\gamma}{2}t}[\sin(\omega_1 t + \phi) + \frac{1}{2}\left(\frac{\gamma}{\omega_1}\right)\cos(\omega_1 t + \phi)]$.
- If the motion is lightly damped then $\frac{\gamma}{\omega_1} \ll 1$, thus $v(t) \simeq -\omega_1 Ae^{-\frac{\gamma t}{2}}\sin(\omega_1 t + \phi)$
- Hence, $K(t) = \frac{1}{2}mv^2 = \frac{1}{2}m\omega_1^2 A^2 e^{-\gamma t}\sin^2(\omega_1 t + \phi)$.
- The potential energy, $U(t) = \frac{1}{2}kx^2 = \frac{1}{2}kA^2 e^{-\gamma t}\cos^2(\omega_1 t + \phi)$
- Therefore, the total energy $E(t) = \frac{1}{2}A^2 e^{-\gamma t}[m\omega_1^2 \sin^2(\omega_1 t + \phi) + k\cos^2(\omega_1 t + \phi)] = \frac{1}{2}kA^2 e^{-\gamma t}$.
- At $t = 0$, energy of the system is $E_0 = \frac{1}{2}kA^2$. Thus, $E(t) = E_0 e^{-\gamma t}$
- This implies exponential decrease of energy in time.

Damped Harmonic Oscillator - V

Q of an Oscillator

- The degree of damping of an oscillator is often specified by a dimensionless parameter Q, known as quality factor.
- $Q = \frac{energy\ strored\ in\ the\ oscillator}{energy\ dissipated\ per\ radian}$
- In light damping case, $\frac{dE}{dt} = -\gamma E_0 e^{-\gamma t} = -\gamma E$
- Hence, energy dissipated in a short time Δt is $\Delta E \sim \left|\frac{dE}{dt}\right| \Delta t = \gamma E \Delta t$
- One radian of oscillation requires time $\Delta t = \frac{1}{\omega_1}$, and the energy dissipated $\frac{\gamma E}{\omega_1}$.
- $Q = \frac{E}{\gamma E/\omega_1} = \frac{\omega_1}{\gamma} \simeq \frac{\omega}{\gamma}$.
- A lightly damped oscillator has $Q \gg 1$ where Q is low for heavily damped oscillator.
- An undamped oscillator has infinite Q.

Example

A paper weight suspended from a hefty rubber band has period of $1.2\ s$ and the amplitude of oscillation decreased by a factor of 2 after three periods. Determine Q of this system.

From the theory of damped oscillator, we know that the amplitude can be expressed as, $Ae^{-\frac{\gamma}{2}t}$.

The ratio of the amplitude at $t = 0$ after three periods (when $t = 1.2 \times 3\ s = 3.6\ s$) is,

$$2 = \frac{Ae^0}{Ae^{-\frac{3.6\gamma}{2}}} \Rightarrow 1.8\gamma = \ln 2 = 0.69 \Rightarrow \gamma = 0.39s^{-1}.$$

Now, $\omega = \frac{2\pi}{T} = \frac{2\pi}{1.2} = 5.23\ s^{-1}$.

Hence, $Q = \frac{\omega}{\gamma} = 13.42$