Coupled Pendulum - I

- Couple : Two things that are linked together, or influence each other.
- Coupled pendulum: Two pendulums linked with each other

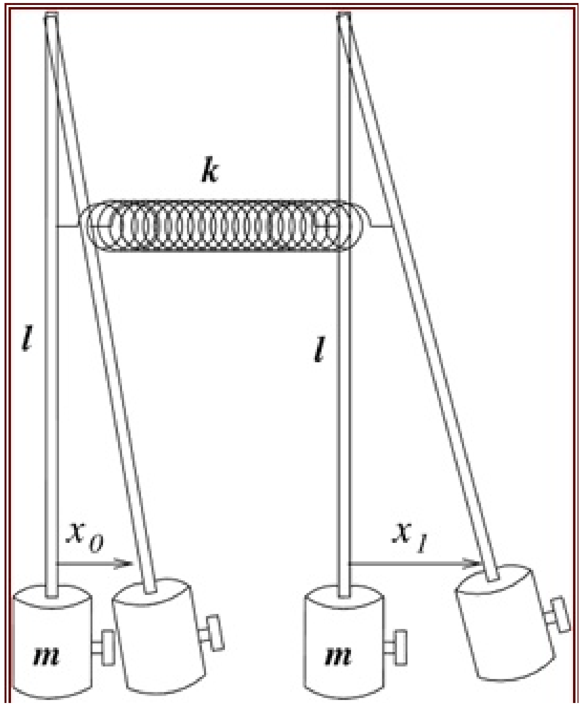
$$F = -kx = m\frac{d^2x}{dt^2} \quad \Rightarrow \quad [k] = \frac{Force}{x} = \frac{mg}{l}$$

- Pendulum 1: $m\frac{d^2x_1}{dt^2} = -\frac{mg}{l}x_1$ Pendulum 2: $m\frac{d^2x_2}{dt^2} = -\frac{mg}{l}x_2$

- Link two pendulums by a spring of spring constant *k.* Introduce a coupling term that will make one pendulum to depend on other's coordinate:

$$m\frac{d^2x_1}{dt^2} = -\frac{mg}{l}x_1 - k(x_1 - x_2)$$

$$m\frac{d^2x_2}{dt^2} = -\frac{mg}{l}x_2 + k(x_1 - x_2)$$



- Coupling force depends on extension of the linking spring, the creates force in opposite directions on the pendulums.

Coupled Pendulum - II

- To solve the coupled equations, we transform variables:

$$X = x_1 + x_2, \qquad Y = x_1 - x_2, \qquad w_0^2 = \frac{g}{l}$$

- Adding the equation of motion of two pendulums, we get,

$$\ddot{x}_1 + \ddot{x}_2 + w_0^2 (x_1 + x_2) = 0 \qquad \Rightarrow \ddot{X} + w_0^2 X = 0$$

- Subtracting the equations we get, $\ddot{Y} + \left(\omega_0^2 + \frac{2k}{m}\right) Y = 0$

Note: Each of the equations in terms of new coordinates X & Y are also SHM.

Normal modes of vibration: A vibration involving only one dependent variable X (or Y) is called a normal mode of vibration and has its own normal frequency. In such a normal mode all components of the system oscillate with the same normal frequency.

The importance of the normal modes of vibration is that they are entirely independent of each other.

The energy associated with a normal mode is never exchanged with another mode; we can add the energies of the separate modes to get the total energy.

Coupled Pendulum - III

In phase mode of vibration:

If $Y = 0$, $x = y$ at all times, so that the motion is completely described by the equation $\ddot{X} + \omega_0^2 X = 0$,

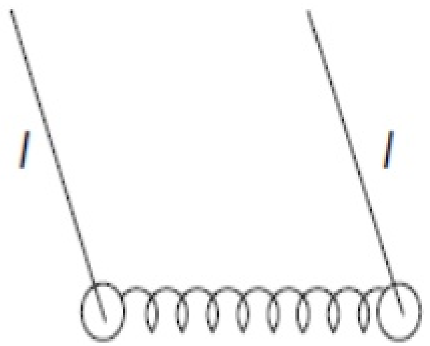
The frequency of oscillation is the same as that of either pendulum in isolation and the stiffness of the coupling has no effect.

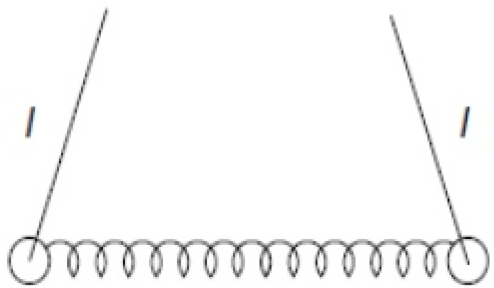
Out of phase mode of vibration:

If $X = 0$, $x = -y$ at all times, so that the motion is completely described by

$$\ddot{Y} + \left(\omega_0^2 + \frac{2k}{m}\right) Y = 0.$$

The frequency of oscillation is greater because the pendulums are always out of phase so that the spring is either extended or compressed and the coupling is effective.

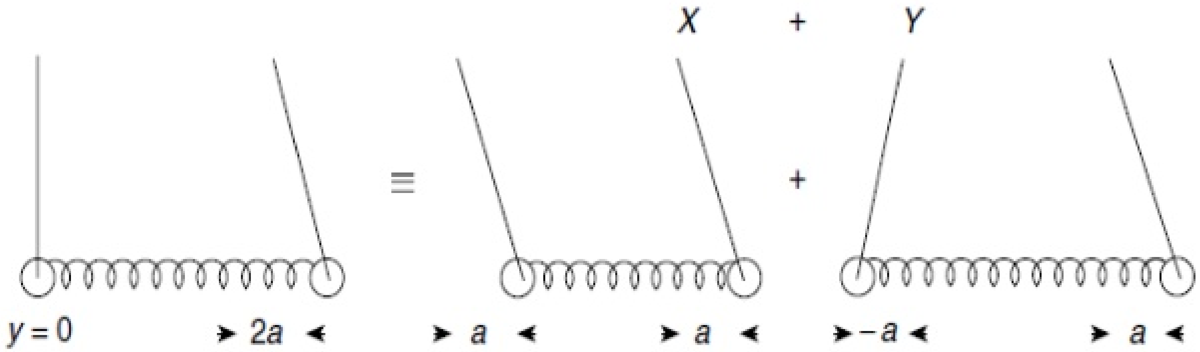




Coupled Pendulum - IV

Any configuration of the coupled system can be represented by the superposition of the two normal modes.

$X = x + y = X_0 \cos(\omega_1 t + \phi_1)$; $Y = x - y = Y_0 \cos(\omega_2 t + \phi_2)$, where X_0 and Y_0 are normal mode amplitudes, and $\omega_1^2 = \frac{g}{l}$ and $\omega_2^2 = \frac{g}{l} + \frac{2k}{m}$.



- **Beats:** Now let us set the system in motion by displacing the right hand mass a distance $x = 2a$ and releasing both the masses from rest so that $\dot{x} = \dot{y} = 0$ at time $t = 0$.
- The initial displacement $x = 2a$, $y = 0$ at $t = 0$ can be seen as a combination of the 'in phase' mode ($x = y = a$ so that $x + y = X_0 = 2a$) and the 'out of phase' mode ($x = -y = a$ so that $Y_0 = 2a$).

Coupled Pendulum - V

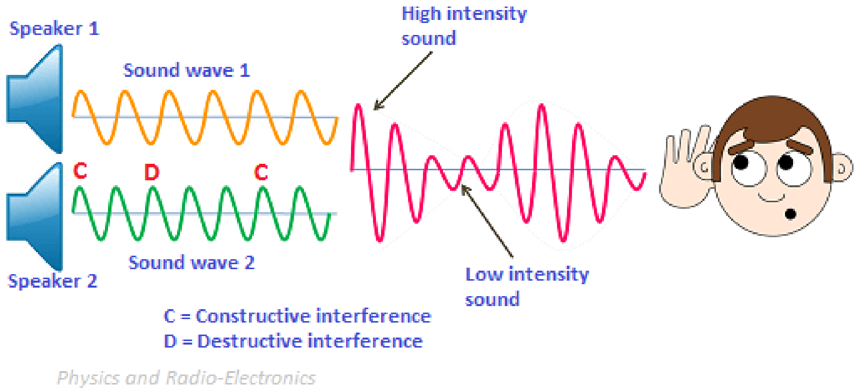
- After release, the motion of the right-hand pendulum is given by

$$x = a\cos(\omega_1 t) + a\cos(\omega_2 t)$$
$$= 2a\cos\frac{(\omega_2-\omega_1)t}{2}\cos\frac{(\omega_1+\omega_2)t}{2}$$

- And that of the left-hand pendulum is given by

$$y = a\cos(\omega_1 t) - a\cos(\omega_2 t)$$
$$= 2a\sin\frac{(\omega_2-\omega_1)t}{2}\sin\frac{(\omega_1+\omega_2)t}{2} .$$

- 'x' and 'y' oscillate cosinusoidally and sinusoidally respectively with a frequency that is the average of the two normal mode frequencies,
- Their amplitudes oscillate with a frequency that is half the difference between the normal frequencies (beats).



Central force problem

Question: What is a central force?

Answer: Any force which is directed towards a center, and depends only on the distance between the center and the particle in question.

Question: Any examples of central forces in nature?

Answer: Two fundamental forces of nature, gravitation, and Coulomb forces are central forces

Question: But gravitation and Coulomb forces are two body forces, how could they be central?

Answer: These two forces are indeed two-body forces, but they can be reduced to central forces by a mathematical trick.

Kepler's law

- Law 1: Every planet moves in an elliptical orbit, with sun on one of its foci.

- Law 2: Position vector of the planet with respect to the sun, sweeps equal areas in equal times.

- Law 3: If T is the time for completing one revolution around sun, and A is the length of major axis of the ellipse, then $T^2 \propto A^3$.

- We aim to derive all these three laws based upon the mathematical theory we develop for central force motion.

Two-body problem

- Gravitational force acting on mass m_1 due to mass m_2 is $\vec{F}_{12} = -\frac{Gm_1m_2}{r_{12}^2}\,\hat{r}_{12}$
- It acts along the line joining two masses.
- Similarly, Coulomb force between two charges q_1 and q_2 is

$$\vec{F}_{12} = \frac{q_1 q_2}{4\pi\epsilon_0 r_{12}^2}\,\hat{r}_{12}$$

- An idea central force can be noted as $\vec{F}(r) = f(r)\hat{r}$
- It is a one-body force depending on the coordinates of only the particle on which it acts.
- But gravity and Coulomb forces are two-body forces, of the form $\vec{F}(r_{12}) = f(r_{12})\hat{r}_{12}$.
- It is now necessary to reduce the two-body problem to a one-body problem.